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On $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Performance of 2D Systems ⋆

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Abstract: In this paper we consider parameterized Lyapunov inequalities arising in $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance evaluation for continuous-discrete 2D systems. A way of transforming these problems is presented that significantly reduces computational complexity compared to known methods. The $\mathcal{H}_\infty$ performance evaluation method can have varying degrees of conservativeness, while $\mathcal{H}_2$ performance evaluation is exact. Numerical examples are provided.

Keywords: linear systems, 2D systems, performance analysis, global optimization, Lyapunov methods, nonlinear programming, matrix inequalities, polynomial inequalities.

1. INTRODUCTION

Multidimensional (nD) systems, in particular, the 2D case, have been receiving much attention in recent years. Examples of such systems include distributed parameter systems (see Cichy et al. (2008); Dullerud and D'Andrea (2004); Rabenstein and Trautmann (2003)), disturbance propagation (see Knorn and Middleton (2013); Li et al. (2005)), etc. Among various interpretations and realizations of 2D systems are repetitive processes (see Rogers et al. (2007); Rogers and Owens (1992)): systems whose independent variables have temporal semantics with one variable describing time as if in a conventional (1D) dynamic system, and another specifying the number of an iteration. The mental model of such a system is a sequence of trajectories of a fixed length constructed in such a way that the output variable as a function of (conventional) time would approach the desired profile. The combined 2D system thus obtained essentially encodes an iterative learning problem (Paszke (2005); Hladowski et al. (2010)). For example, teaching the continuous system

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t), \quad y_k(t) = Cx_k(t),$$

to follow the desired trajectory $y_d(t)$ on $t \in [0;T]$ for a given T, can be represented as a 2D system

$$\left[\, \dot{\eta}_k(t) \;\; e_{k+1}(t) \,\right]^{\mathrm{T}} = M \left[\, \eta_k(t) \;\; e_k(t) \,\right]^{\mathrm{T}},$$

where $e_k(t) = y_d(t) - y_k(t)$ is the tracking error, $\eta_k(t) = \int_0^t x_{k+1}(\tau) - x_k(\tau)\,\mathrm{d}\tau$, and M depends on A, B, C, and coefficient matrices defining the control law. Convergence of the learning process in this setup is equivalent to the 2D system's stability. Applications of these can be found in fields like chemical batch processes, metal rolling, or coal mining (see the references above).

Analysis of 2D systems is, in a way, similar to more conventional dynamic system analysis, e.g., stability checking and performance estimation. However, complexity of tests and computational burden are significantly higher.

There are various approaches to analysis of 2D systems. They are generally based on characteristic polynomials or multinomials (see, e.g., Agathoklis et al. (1993) for the double-discrete case, and Rogers and Owens (2002) for continuous-discrete time systems), or specialized Lyapunov functions. The latter allow using techniques based on linear matrix inequalities (LMIs) and convex optimization. They also support more complex types of analysis/synthesis problems (see, e.g., Paszke et al. (2011)). An interesting method—which forms a basis for this paper's results—is to find a way to extend LMI systems used for 1D systems' stability and performance analysis to make them applicable to 2D systems. This (nontrivial) process yields problems of determining positivity of matrix polynomials that can be treated as sum-of-squares (SOS) problems and ultimately recast in semidefinite programming forms. A uniform treatment of 2D system stability and $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance estimation can be found in Chesi and Middleton (2014, 2015). Further details are provided in section 2.1.

Approaches based on Lyapunov functions and generally matrix inequalities can often provide necessary and sufficient conditions in terms of polynomial or linear matrix inequalities (PMIs and LMIs, respectively). However, when such conditions are obtained, they are not always practical to check due to their numerical complexity. This issue can be partially solved by, e.g., replacing PMIs with their approximations. This introduces a degree of conservativeness, but a proper strategy of constructing sequences, or hierarchies, of such approximations may result in eventually necessary and sufficient conditions. See Chesi and Middleton (2014); Henrion and Lasserre (2006) for examples of such sequences.

Section 2 of this paper presents the problems and relevant background information from existing works. A possible way—proposed earlier by the author—to tackle the numerical complexity issue is described in section 3. The main contribution of this paper is provided in sections 4 and 5; section 6 shows numerical examples.

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2. BACKGROUND

2.1 2D Systems

2D systems have a number of representations. In this paper, we consider 2D mixed continuous-discrete-time repetitive processes that can be written down as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x_c(t,k) &= A_{cc}x_c(t,k) + A_{cd}x_d(t,k) + B_c u(t,k), \\ x_d(t,k+1) &= A_{dc}x_c(t,k) + A_{dd}x_d(t,k) + B_d u(t,k), \\ y(t,k) &= C_c x_c(t,k) + C_d x_d(t,k) + Du(t,k), \end{aligned} \tag{1}$$

where $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$ are the continuous and discrete time variables, $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, and the rest of notation keep their general meaning.

For $\mathcal{H}_\infty$ performance analysis of (1), the paper Chesi and Middleton (2014) proposes a method for finding upper bounds $\hat{\gamma}_\infty(2d)$ of the $\mathcal{H}_\infty$ performance

$$\gamma_\infty = \sup_{\omega,\theta} \left\| Q(e^{j\theta}, j\omega) \right\|_2,$$

where $Q(z,s) \in \mathbb{C}^{n_y \times n_u}$ is the transfer function from the Laplace-Z transform of $u(t,k)$ to the Laplace transform of $y(t,k)$:

$$\begin{aligned} Q(z,s) &= F_3(s)(zI - F_1(s))^{-1}F_2(s) + F_4(s), \\ F_1(s) &= A_{dc}(sI - A_{cc})^{-1}A_{cd} + A_{dd}, \\ F_2(s) &= A_{dc}(sI - A_{cc})^{-1}B_c + B_d, \end{aligned}$$

$$\begin{aligned} F_3(s) &= C_c(sI - A_{cc})^{-1}A_{cd} + C_d, \\ F_4(s) &= C_c(sI - A_{cc})^{-1}B_c + D. \end{aligned}$$

These bounds are calculated as $\hat{\gamma}_\infty(2d) = \sqrt{\hat{\xi}(2d)}$, where $\hat{\xi}(2d)$ is a solution to the following problem:

$$\begin{aligned} &\hat{\xi}(2d) = \inf_{P,\xi,c} \xi, \\ &\forall \omega \in \mathbb{R} : V(\xi,\omega) - c\left|g(j\omega)\right|^2 I \geq 0, \\ &c > 0, \\ &\deg(P(\omega)) = 2d, \end{aligned} \tag{2}$$

where $\xi, c \in \mathbb{R}$; $P(\omega) \in \mathbb{C}^{n_d \times n_d}$ is a Hermitian matrix polynomial of a pre-chosen degree $2d$; $g(s) = \det(sI - A_{cc})$; and

$$V(\xi,\omega) = |g(j\omega)|^2 \begin{pmatrix} P(\omega) & F_1(j\omega)P(\omega) & F_2(j\omega) & 0 \\ * & P(\omega) & 0 & P(\omega)F_3(j\omega)^H \\ * & * & I & F_4(j\omega)^H \\ * & * & * & \xi I \end{pmatrix}.$$

Note that $V(\xi,\omega)$ and the left hand side of the matrix inequality constraint in the above system are matrix polynomials, not rational functions. Conservatism of these bounds is directly dependent on the degree $2d$ of $P(\omega)$. A theorem is provided for determining whether the bound is tight ($\gamma_\infty = \hat{\gamma}_\infty(2d)$).

Finding $\mathcal{H}_2$ performance of a stable system (1) is described in Chesi and Middleton (2015) using a computational scheme somewhat similar to the above. We need to choose an upper bound $2d$ for an expected degree of a Hermitian matrix polynomial $W(\omega)$. After that, we look for $W(\omega)$ and ϵ such that

$$\begin{aligned} &\forall \omega \in \mathbb{R} : \left|g(j\omega)\right|^2 (W(\omega) - F_1(\omega)W(\omega)F_1(\omega)^H - \\ &\quad - v(\omega)F_2(\omega)F_2(\omega)^H - \epsilon v(\omega)I) \geq 0, \\ &\forall \omega \in \mathbb{R} : W(\omega) - \epsilon v(\omega)I \geq 0, \\ &\epsilon > 0, \\ &\deg(W(\omega)) \leq 2d - 2, \end{aligned}$$

where $v(\omega) = (1+\omega^2)^d$. $\mathcal{H}_2$ performance γ_2 does not exceed $\sqrt{\zeta}$ with $\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\omega)\,\mathrm{d}\omega$,

$$\begin{aligned} \phi(\omega) &= \mathrm{tr}(F_3(\omega)W_{RAT}(\omega)F_3(\omega)^H + F_4(\omega)F_4(\omega)^H), \\ W_{RAT}(\omega) &= W(\omega)/v(\omega). \end{aligned}$$

The paper also specifies a way to estimate deviation of the discovered bound from the actual performance value.

2.2 Polynomial Matrix Inequality Systems

Consider the problem of finding global extrema of a polynomial function over a region defined by PMIs:

$$\begin{aligned} &f^* = \min_x f(x), \\ &G_i(x) \geq 0, \\ &x \in \mathbb{R}^n, \quad G_i(x) = G_i^{\mathrm{T}}(x) \in \mathbb{R}^{n_i \times n_i}, \quad i = 1,\dots,m, \end{aligned} \tag{3}$$

where $f(x)$ and elements of $G_i(x)$ are (not necessarily convex) polynomials, and the inequality sign in (3) denotes positive semidefiniteness. Hereafter, these problems will be called *PMI problems*.

In Henrion and Lasserre (2005, 2006); Lasserre (2001), a solution method for this kind of problems has been proposed. It was based on constructing a hierarchy of LMI relaxations that would approximate the original problem in the space of its indeterminates' moments. An LMI relaxation in this context is a system of the following kind:

$$\begin{aligned} &f^* = \min_y \sum_i f_i y_i, \\ &M_k(y) \geq 0, \\ &M_{k-d_i}(G_i, y) \geq 0, \quad i = 1,\dots,m, \\ &y_1 = 1, \end{aligned}$$

where k is the relaxation order; $d_i = \lceil \frac{1}{2}\deg G_i(x) \rceil$; $y = [y_i]_i = \int b_{2k}(x)\,\mathrm{d}\mu$ is the vector of moments of some unknown measure μ; $b_r(x)$ is the monomial basis of the space of polynomials having degrees up to r:
$b_r(x) = [1\ x_1\ x_2\ \dots\ x_n\ x_1^2\ x_1x_2\ \dots\ x_n^2\ \dots\ x_1^r\ \dots\ x_n^r]^{\mathrm{T}}$;
vector $[f_i]_i$ is a representation of $f(x)$ in this basis: $f(x) \equiv \sum_i (b_k(x))_i f_i$. $M_k(y)$ and $M_{k-d_i}(G_i, y)$ are the moment matrix and localizing matrices derived from

$$M_k(y) \equiv \int b_k(x)b_k(x)^{\mathrm{T}}\,\mathrm{d}\mu,$$

$$M_{k-d}(G,y) \equiv \int \left(b_{k-d}(x)b_{k-d}(x)^{\mathrm{T}}\right) \otimes G(x)\,\mathrm{d}\mu.$$

For $k \to \infty$, the minimum of the LMI relaxation approaches the minimum of the original PMI problem; in practice, for many problems they become equal for finite, relatively small, values of k, and the vector of moments of PMI solution becomes a solution to the LMI relaxation.

3. ATOMIC OPTIMIZATION

The global optimization method described in section 2.2 is quite flexible and powerful. On the other hand, it suffers