Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-13 (2016) 048–052

On Lower Bound of Anisotropic Norm ⋆

Victor A. Boichenko * **Alexander P. Kurdyukov** *

* *V.A. Trapeznikov Institute of Control Sciences, RAS, 65 Profsoyuznaya Str., 117997 Moscow GSP-4, Russia (e-mail: v.boichenko@gmail.com, akurdyukov2010@mail.ru)*

Abstract: The anisotropic norm of a linear discrete time invariant system is a measure of system output sensitivity to stationary Gaussian input disturbances with mean anisotropy bounded by some nonnegative parameter. The mean anisotropy characterizes the predictability (or coloredness) degree of stochastic signal. The anisotropic norm of a system is an induced norm, limiting cases of which are $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms as $a \to 0$ and $a \to \infty$, respectively. In Vladimirov et al. (1996) a method for numerical computation of the anisotropic norm was proposed. This method involves cross-coupled Riccati and Lyapunov equations and associated special type equation. Another method for anisotropic norm computation via LMI-based approach was proposed in Tchaikovsky & Kurdyukov (2006). This paper develops a new method for getting the lower bound for anisotropic norm by linear algebra standard methods.

Keywords: anisotropic norm, Lyapunov equation, $\mathcal{H}_2$-norm, $\mathcal{H}_\infty$-norm

1. INTRODUCTION

The well known $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimization approaches for linear time-invariant control systems are based on using $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms in the respective Hardy spaces of matrix-valued transfer functions. The performance criteria applied in these approaches are determined by different assumptions about the nature of input disturbances affecting the system.

The principal assumption of $\mathcal{H}_2$-theory posits that the input disturbance is the Gaussian white noise signal. $\mathcal{H}_\infty$-theory treats the plant modelling errors as a norm-bounded perturbation of a nominal transfer function and input disturbances as square-summable signals.

A stochastic approach to $\mathcal{H}_\infty$-optimization of discrete time invariant control systems is based on using a stochastic norm in performance criteria. The stochastic norm measures the sensitivity of the system output to random input disturbances, probability distribution of which is not precisely known.

According to Semyonov et al. (1994), Diamond et al. (2001), the a-anisotropic norm of a system is a particular case of the stochastic norm and is defined as the supremum of the ratio of the root mean square value of the system output to that of the input over all stationary Gaussian inputs with the mean anisotropy upper-bounded by a nonnegative parameter a.

For the absolutely continuously distributed Gaussian random vector, the anisotropy is defined as a difference between the differential entropy of the Gaussian random vector with zero mean and constant diagonal covariance matrix and the differential entropy of this vector, and can be considered as a measure of distinction between the covariance matrix of a random vector and the identity matrix; see Diamond et al. (2001). This quantity is always nonnegative. Considering finite but arbitrarily long initial segments of the stationary Gaussian sequence as such vectors, one can define the anisotropy per unit of time and the mean anisotropy as its limiting value. The mean anisotropy is an entropy-based measure of predictability (or coloredness) of the Gaussian signal.

For any fixed mean anisotropy level $a > 0$, the a-anisotropic norm lies between the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms and induces an intermediate topology, weaker than the C-topology and stronger than the respective h-topology. Moreover, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of a system are the limiting cases of the a-anisotropic norm as $a \to 0$ and $a \to \infty$, correspondingly. The 0-anisotropic norm coincides with the $\mathcal{H}_2$-norm up to a constant multiplier depending on the dimension of the system input.

The method for anisotropic norm calculation involves the solving of cross-coupled algebraic Riccati equation, Lyapunov equation and special type equation. Sometimes it is necessary to know only the lower bound for anisotropic norm. Such problem arises in the anisotropy-based adaptive control. This paper presents a new approach to determining the lower bound for anisotropic norm by standard methods of the linear algebra.

The paper is organized as follows. Section 2 recalls the minimum necessary background on the anisotropy of signals and the anisotropic norm of systems. Section 3 establishes the main result. In Section 4 the anisotropic gain of a system with a two-dimensional input is calculated. Section 5 provides an illustrative numerical example. Concluding remarks are given in Section 6.

⋆ The paper is supported by the Russian Foundation for Basic Research grant 14-08-00069.

2405-8963 © 2016, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.07.925

2. BASIC CONCEPTS OF ANISOTROPY-BASED ROBUST PERFORMANCE ANALYSIS

Let us recall a minimum necessary background material on the anisotropy of signals and anisotropic norm of systems. Full information on the anisotropy-based robust performance analysis developed originally by Vladimirov et al. (1995, 1996) can be found in more detail in Diamond et al. (2001), Vladimirov et al. (2006).

Let $\mathbb{L}_2^m$ denote the class of square integrable $\mathbb{R}^m$-valued random vectors distributed absolutely continuously with respect to the m-dimensional Lebesgue measure mes_m. For any $w \in \mathbb{L}_2^m$ with probability density function $f\colon \mathbb{R}^m \to \mathbb{R}_+$, the *anisotropy* $\mathbf{A}(w)$ is defined by Vladimirov et al. (2006) as the minimal value of relative entropy $\mathbf{D}(f\|p_{m,\lambda})$ with respect to the Gaussian distributions $p_{m,\lambda}$ in $\mathbb{R}^m$ with zero mean and scalar covariance matrices λI_m:

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{D}(f\|p_{m,\lambda}) = \frac{m}{2}\ln\left(\frac{2\pi\,\mathrm{e}}{m}\mathbf{E}[|w|^2]\right) - \mathbf{h}(w), \tag{1}$$

where $\mathbf{E}[\cdot]$ is an expectation and $\mathbf{h}(w)$ denotes the differential entropy of w with respect to mes_m; see Cover & Thomas (1991). The minimum in (1) is achieved at $\lambda = \mathbf{E}[|w|^2]/m$; see Vladimirov et al. (2006).

Let $W = \{w_k\}_{-\infty<k<+\infty}$ be a stationary sequence of vectors $w_k \in \mathbb{L}_2^m$ interpreted as a discrete-time random signal. Assemble the elements of W associated with a time interval $[s,t]$ into a random vector

$$W_{s:t} = \mathrm{col}\,[w_s \cdots w_t]. \tag{2}$$

It is assumed that $W_{0:N}$ is distributed absolutely continuously for every $N \geqslant 0$. The *mean anisotropy* of the sequence W is defined by Vladimirov et al. (2006) as the anisotropy production rate per time step by

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{3}$$

Let $\mathbb{G}^m(\Sigma)$ denote the class of $\mathbb{R}^m$-valued Gaussian random vectors with zero mean $\mathbf{E}[w_k] = 0$ and nonsingular covariance matrix $\mathbf{cov}(w_k) = \mathbf{E}[w_k\, w_k^{\mathrm{T}}] = \Sigma$. Let $V = \{v_k\}_{-\infty<k<+\infty}$ be a sequence of random vectors $v_k \in \mathbb{G}^m(I_m)$, i.e. the m-dimensional Gaussian white noise sequence. Suppose $W = GV$ is produced from V by a stable shaping filter with the transfer function $G(z) \in \mathcal{H}_2^{m\times m}$. Then the spectral density of W is given by

$$S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega), \quad -\pi \leqslant \omega < \pi, \tag{4}$$

where $\widehat{G}(\omega) = \lim_{r\to 1-} G(r\mathrm{e}^{i\omega})$ is the boundary value of the transfer function $G(z)$. As is shown by Vladimirov et al. (1996), Diamond et al. (2001), the mean anisotropy (3) can be computed in terms of spectral density (4) and the associated $\mathcal{H}_2$-norm of the shaping filter G as

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega. \tag{5}$$

Since the probability law of the sequence W is completely determined by the shaping filter G, the alternative notation $\overline{\mathbf{A}}(G)$ is also used instead of $\overline{\mathbf{A}}(W)$.

The mean anisotropy functional (5) is always nonnegative. It takes a finite value if the shaping filter G is of full rank, otherwise $\overline{\mathbf{A}}(G) = +\infty$; see Vladimirov et al. (1996), Diamond et al. (2001). The equality $\overline{\mathbf{A}}(G) = 0$ holds true if and only if G is an all-pass system up to a nonzero constant factor. In this case, the spectral density (4) is described by $S(\omega) = \lambda I_m$, $-\pi \leqslant \omega < \pi$, for some $\lambda > 0$, so that W is a Gaussian white noise sequence with zero mean and a scalar covariance matrix.

Given the stationary Gaussian sequence $W = GV$, generated by a filter $G \in \mathcal{H}_2^{m\times m}$, consider the sequences $\widehat{W} = \{\widehat{w}_k\}_{-\infty<k<+\infty}$ and $\widetilde{W} = \{\widetilde{w}_k\}_{-\infty<k<+\infty}$ of predictors and prediction errors defined by

$$\widehat{w}_k = \mathbf{E}[w_k|\{w_j\}_{j<k}], \quad \widetilde{w}_k = w_k - \widehat{w}_k, \tag{6}$$

where $\mathbf{E}[\cdot|\cdot]$ denotes the conditional expectation. Since $\mathbf{E}[\widetilde{w}_k|\{w_j\}_{j<k}] = 0$, the sequence $\widetilde{W}$ is the Gaussian white noise in general not with identity covariance matrix.

Lemma 1. The mean anisotropy (5) of the sequence $W = GV$ generated by a maximal rank filter $G \in \mathcal{H}_2^{m\times m}$ is expressed in terms of the second-order moments of W and $\widetilde{W}$ as

$$\overline{\mathbf{A}}(G) = -\frac{1}{2}\ln\det\left(\frac{m\,\mathbf{E}[\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}]}{\mathbf{E}\left[|w_0|^2\right]}\right). \tag{7}$$

The proof of this lemma can be found in Diamond et al. (2001). The representation (7) affords to split the mean anisotropy into *temporal* and *spatial* parts

$$\overline{\mathbf{A}}(G) = \overline{\mathbf{A}}_t(G) + \overline{\mathbf{A}}_s(G), \tag{8}$$

where

$$\overline{\mathbf{A}}_t(G) = \frac{1}{2}\ln\det\left(\mathbf{E}[w_0w_0^{\mathrm{T}}]\left(\mathbf{E}[\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}]\right)^{-1}\right), \tag{9}$$

$$\overline{\mathbf{A}}_s(G) = -\frac{1}{2}\ln\det\left(\frac{m\,\mathbf{E}[w_0w_0^{\mathrm{T}}]}{\mathbf{E}\left[|w_0|^2\right]}\right). \tag{10}$$

By the probability structure of the sequences (6), the temporal term $\overline{\mathbf{A}}_t(G)$ coincides with the information about w_0 contained in the past history $\{w_k\}_{k<0}$ of the sequence $W = GV$

$$\overline{\mathbf{A}}_t(G) = \mathbf{I}(w_0,\, \{w_k\}_{k<0}) \tag{11}$$

and therefore it quantifies predictability (or coloredness) of the signal.

Let $F \in \mathcal{H}_\infty^{p\times m}$ be a linear discrete time invariant (LDTI) system with an m-dimensional input W and a p-dimensional output $Z = FW$. Let the random input sequence $W = GV$, where as before $V \in \mathbb{G}^m(I_m)$. Denote by

$$\mathcal{G}_a = \left\{G \in \mathcal{H}_2^{m\times m} : \overline{\mathbf{A}}(G) \leqslant a\right\} \tag{12}$$

the set of shaping filters G that produce Gaussian random sequences W with mean anisotropy (5) bounded by a given parameter $a \geqslant 0$.

The *a-anisotropic norm* of the system F is defined as

$$|\!|\!| F |\!|\!|_a = \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{13}$$

see Vladimirov et al. (1996), Diamond et al. (2001).

It is shown by Vladimirov et al. (1996) that the a-anisotropic norm of a given system $F \in \mathcal{H}_\infty^{p\times m}$ is a nondecreasing continuous function of the mean anisotropy level a, which satisfies

$$\frac{1}{\sqrt{m}}\|F\|_2 = |\!|\!| F |\!|\!|_0 \leqslant |\!|\!| F |\!|\!|_a \leqslant \lim_{a\to+\infty} |\!|\!| F |\!|\!|_a = \|F\|_\infty. \tag{14}$$