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On the Anisotropy-Based Bounded Real Lemma Formulation for the Systems with Disturbance-Term Multiplicative Noise ⋆

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Abstract: This paper is an attempt to present the methodologically correct approach of the anisotropy-based theory for discrete-time systems with multiplicative noise. A formulation of sufficient conditions of boundedness of the anisotropic norm is the main goal of this paper. As it is shown, this problem can be reduced to the convex optimization problem. A simple numerical example to illustrate the solution procedure is given.

Keywords: bounded real lemma, stochastic control systems, anisotropic norm, multiplicative noise

1. INTRODUCTION

One of the popular control design method from the beginning of the 60-s is LQG control design minimizing $\mathcal{H}_2$-norm of closed-loop system, see Kalman (1960). It is proposed that the input of the system is the Gaussian white noise with zero mean value and identity covariance matrix.

During last 30 years, control design methods minimizing $\mathcal{H}_\infty$-norm of the closed-loop system transfer function from square-integrable external disturbance to controllable output have successfully being applied, see Doyle et al. (1989).

As it is known from Doyle (1978), the LQG controller performs poorly if the input disturbance is strongly colored noise, while $\mathcal{H}_\infty$-controller designed for the worst case disturbance demonstrates excessive conservatism if the external disturbance is white noise or weakly colored noise.

The anisotropy-based stochastic robust control theory for linear discrete time-invariant (LDTI) systems has being developed from the beginning of the 90-s in Russia and other countries, see Vladimirov et al. (1995a), Vladimirov et al. (1995b). The main concept of the anisotropy-based approach to robust stochastic control is anisotropy (the distinction-like function) of a random vector. Anisotropy is defined as the minimum value of Kullback-Leibler divergence with respect to a scalar parameter λ. The important notion in the anisotropy-based theory belongs to the concept of mean anisotropy of a stationary ergodic sequence of random vectors, Vladimirov et al. (1995b). The performance index of a system is anisotropic norm. According to mean anisotropy of input sequence, the anisotropic norm takes values between scaled $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm of a system. The convergence rate of anisotropic norm of a system to its limiting values (scaled $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm) is investigated in Vladimirov et al. (1999), Vladimirov et al. (2006).

Mean anisotropy of random vector equals 0 for the Gaussian white noise, and tends to ∞ when input is deterministic square-integrable sequence. It means that $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control problems are the limiting cases of anisotropy-based control theory, see Diamond et al. (2001).

A state-space solution of the anisotropy-based optimal control problem results in the unique full-order estimator-based controller and involves the solution of the three cross-coupled algebraic Riccati equations, the algebraic Lyapunov equation and one special-type equation, see Vladimirov et al. (1996). To solve this system of equations, application of the homotopy numerical algorithm is required, see Diamond et al. (1997).

The anisotropy-based suboptimal controller design is a natural extension of this approach. Instead of minimizing anisotropic norm of the closed-loop system, a suboptimal controller is required to keep it below a given threshold value. Rather than resulting in the unique controller, the suboptimal design yields the family of controllers, thus providing freedom to impose some additional performance specifications to the closed-loop system. The anisotropy-based suboptimal control design requires a state-space criterion for verifying if the anisotropic norm of a system does not exceed a given value. The Anisotropy-Based Bounded Real Lemma (ABBRL) is the stochastic counterpart of the $\mathcal{H}_\infty$ Bounded Real Lemma for LDTI systems with statistically uncertain Gaussian disturbances with bounded mean anisotropy, see Kurdyukov et al. (2010). The resulting criterion has the form of the inequality on the determinant of the matrix associated with the algebraic Riccati equation. These results are applied to design of the suboptimal anisotropic controllers by means of convex optimization

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and semi-definite programming, see Tchaikovsky et al. (2011a)

The anisotropy-based theory was successfully extended to the descriptor LDTI systems. The solution of this problem was obtained both by solving the Riccati equations and by the methods of convex optimization problems, see respectively Andrianova et al. (2015) and Belov et al. (2015).

Another example of a possible generalization is the theory of stochastic control systems with multiplicative noise. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases of this theory has been studied in Gershon et al. (2005). First attempts to generalize the anisotropy-based control theory for systems with multiplicative noise have been done in Stoica & Yaesh (2012) and Stoica & Yaesh (2014). However, these attempts have in our opinion some disadvantages.

Firstly, the property of mean anisotropy of a sequence of the Gaussian random vectors has been chosen as the definition of mean anisotropy for the non-Gaussian sequence. It seems to be incorrect methodologically. Secondly, the system of nonlinear equations and inequalities has been obtained for getting the boundedness conditions for the anisotropic norm of stochastic system with multiplicative noise. The method for the solving of these nonlinear equations and inequalities is not pointed in Stoica & Yaesh (2014).

The usage of the game approach for solving the problems of anisotropy-based analysis and synthesis (as it was in $\mathcal{H}_\infty$-problem) is not meaningful until formulas of anisotropy of the non-Gaussian random vector and mean anisotropy of a sequence of the non-Gaussian random vectors will be obtained. Using the game approach, all matrices of the worst-case filter (i.e. the solution of the problem) should be depend on the matrices of the initial system. It means that the sequence generated by that filter is not the Gaussian.

In connection with the above, it makes sense to obtain some sufficient conditions for the anisotropic norm boundedness for the systems with multiplicative noise, without giving precise formulas for anisotropy and mean anisotropy. In other words, it makes sense to get sufficient conditions by use of certain majorants.

This paper concerns the obtaining of the anisotropic norm boundedness conditions for the systems with multiplicative noise in disturbance term of system. The preliminaries are given in section 2 . The problem statement and the main result are given in section 3. Some numerical example is given in section 4.

2. BACKGROUND

As it was mentioned above, anisotropy of random vector, mean anisotropy of random vector sequences and anisotropic norm of the system are the basic concepts of anisotropy-based control theory, for more details see Diamond et al. (2001) and Kustov et al. (2010). Let us give the definitions of these basic concepts.

Anisotropy of the Gaussian m-dimensional random vector $w \sim \mathcal{N}(0, \Sigma)$ with probability density function f is defined as

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{D}(f||p_\lambda), \tag{1}$$

where

$$p_\lambda(x) = ((2\pi\lambda))^{-m/2} \exp\left(-\frac{x^T x}{2\lambda}\right), \quad x \in \mathbb{R}^m,$$

is the etalon probability density function, and

$$\mathbf{D}(f||g) = \mathbf{E}_f\left[\ln\left(\frac{f}{g}\right)\right] = \int_{\mathbb{R}^m} f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx$$

is Kullback-Leibler information divergence of f with respect to g. Here $\mathbf{E}_f[\cdot]$ is the mathematical expectation symbol (in the sense of f).

If w is the Gaussian random vector with probability density function

$$f(x) = ((2\pi)^m \det(\Sigma))^{-1/2} \exp\left(-\tfrac{1}{2} x^T \Sigma^{-1} x\right), \quad x \in \mathbb{R}^m,$$

then the "optimal" $\lambda > 0$ in (1) is equal to $\mathrm{tr}\Sigma/m$, and the corresponding probability density function p_λ is the closest one to f in the sense of value of $\mathbf{D}(f||p_\lambda)$. In this case,

$$\mathbf{A}(w) = -\frac{1}{2} \ln\det\left(\frac{m\Sigma}{\mathrm{tr}\Sigma}\right).$$

Now consider the stationary ergodic sequence of the Gaussian random vectors $W = \{w_k\}_{k\in\mathbb{Z}}$. For the vector $W_{0:N} = (w_0^T, \ldots, w_N^T)$ associated with the fragment of the sequence $\{w_k\}$, $k = \overline{0, N}$, the limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N})}{N},$$

exists and is called mean anisotropy of the sequence W.

Let $F \in \mathcal{H}_\infty^{p\times m}$ be LDTI system with m-dimensional input w and p-dimensional output z, and let the property $\|F\|_2/\sqrt{m} < \|F\|_\infty$ holds for this system. Anisotropic norm of the system F is defined as

$$|\!|\!| F |\!|\!|_a = \sup_{W:\overline{\mathbf{A}}(W)\leqslant a} \frac{\|z\|_2}{\|w\|_2}, \tag{2}$$

where $\|\cdot\|_2$ is the stochastic l_2-norm of the signal:

$$\|x\|_2 = \left(\sum_{k=0}^{\infty} \mathbf{E}[|x_k|^2]\right)^{\frac{1}{2}}.$$

One of the most important properties of the anisotropic norm is that it takes values from the following interval:

$$\frac{\|F\|_2}{\sqrt{m}} = \lim_{a\downarrow 0} |\!|\!| F |\!|\!|_a \leqslant |\!|\!| F |\!|\!|_a \leqslant \lim_{a\uparrow\infty} |\!|\!| F |\!|\!|_a = \|F\|_\infty.$$

3. MAIN RESULT

3.1 Problem statement

Consider the system with disturbance-term multiplicative noise

$$F \sim \begin{cases} x_{k+1} = Ax_k + \mathcal{B}w_k, \\ z_k = Cx_k + Dw_k, \end{cases} \tag{3}$$

where $\mathcal{B} = B_0 + \sum_{i=1}^{r} B_i s_{i,k}$ is the input-to-state matrix with the Gaussian random noise, $x_k \in \mathbb{R}^n$ is a state vector, $w_k \in \mathbb{R}^m$ and $z_k \in \mathbb{R}^p$ are input and output vectors respectively. All matrices A, B_i, C, D are known. The sequences $\{s_{i,k}\}_{k\in\mathbb{Z}}$ are the 1-dimensional Gaussian white