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Asymptotically Optimal Solution for Transportation Problem with Almost Arbitrary Capacities

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Abstract: Consider a following transportation problem: given a set of nodes, some pairs of them are connected; each node has some initial quantity of a single-type product and a demand of the same product; the amount of product that can be transferred between connected nodes per time unit is limited; assuming that summary demand equals summary initial quantity of the product, what is the minimal time required to satisfy all the demands? For the case of constant capacities over time for each pair of connected channels, this problem is quite a simple optimization problem (linear programming or simpler) but might become very hard if it is not constant. In this paper we present a proof that for the problem with "averagable" in some sense capacity functions one can construct a suboptimal solution based on the solution of the "averaged" problem that happens to be asymptotically optimal.

Keywords: Transportation problem, optimal control, network optimization, nonlinear optimization.

1. INTRODUCTION

Transportation problem is probably one of the most researched optimization problems. Transportation technologies drastically influence many aspects of humanity evolution in general. In many cases it happens that one can improve utility of a transportation process in large systems simply by a proper adjustment the system parts. The question of "optimal adjustment" thus leads to some optimization problems which are in general simply referred to as transportation problems in mathematics.

Although there are numerous works on transportation problem in the field of mathematical optimization, in this paper we are mostly interested in the transportation models that include some kind of uncertainty. In general it is a trend (in many optimization problem applications) of last few decades not to find an optimal plan for a one particular scenario but rather give a plan that works optimal or near optimal in a large number of possible situations (*e.g.* optimizing the worst-case scenario, optimizing average performance or some other criterion). The necessity of these approaches is typically caused by discrepancy of physical behaviour and mathematical models. It is well-known that mathematical models cannot be exact, but direct inclusion of the "uncertainty" in the model might significantly improve the accuracy.

In the study of transportation problem this issue is addressed in many works including stochastic formulations researched by Mahapatra et al. (2013); Tan et al. (2013); uncertain models in the works of Yang et al. (2015); Gabrel et al. (2014); models with fuzzy constrains Kulak and Kahraman (2005); Lau et al. (2009); Basirzadeh (2011); Giri et al. (2015); Srinivas and Ganeshan (2015) and many other works.

Next, one of the important aspect of transportation problems is temporal dimension. The problems that does not include study of flowing over time processes are usually attributed as static and contrary, dynamic problems imply the study of the flowing over time transportation processes. Detailed overview of dynamic transportation problems is given by Pillac et al. (2013); Ran and Boyce (2012).

Motivation of the current work is focused on the fact that existing techniques of handling the uncertainties does not include varying over time uncertainties, or in other words both dynamic and uncertain aspects rarely appear in the same model, see Pillac et al. (2013).

The contribution of this paper is the study of a flow-typed transportation problem under a time-varying capacity functions. Here we focus only on the aspect of capacities and ignore cost constrains that allows us to obtain some interesting results. In general, proposed method is to solve a rather simple "averaged" problem and adapt the solution for initial problem. In this method some kind of prediction is conducted but on the other hand we try to include uncertainties in this prediction. Thus, we need a model to be regular in some sense. Necessary conditions for such prediction is strictly described by "averagability" which includes the cases of periodic functions and some normal stochastic processes (like Wiener process).

Summary, the main results of the paper are the concept of capacity averaging, proof of its asymptotic optimality (compared to averaged problem) and examples of its possible applications.

The paper is organized as follows: section 2 contains problem formulation; in section 3 the main results are presented, namely class of averagable functions and corresponding theorem that describes suboptimal control for averagable functions; section 4 contains examples of application: possible interpretation of a random process as an averagable function and application to load balancing problem.

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2. PRELIMINARIES

2.1 Problem in study

Suppose there is a directed graph $G = \langle V, E\rangle$, V is a set of nodes / vertices ($|V| = n$) and E is a set of edges / arcs between the nodes, $E \subset V \times V$. Assuming that all the vector inequalities are component-wise inequalities, consider the following transportation problem

$$\begin{aligned} &\textit{minimize} \quad \tau, \\ &\textit{subject to} \begin{cases} \dot{\mathbf{x}} = B\mathbf{u}, \\ \mathbf{x}(0) = \mathbf{x}^-;\ \mathbf{x}(\tau) = \mathbf{x}^+, \\ \mathbf{x}(t) \geq 0, \\ 0 \leq u_{ij}(t) \leq c_{ij}(t). \end{cases} \end{aligned} \tag{1}$$

Here, $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is the state vector (or phase variables), $x_i(t)$ represents the volume of a product of node i at time moment t. $c_{ij}(t)$ represents capacity of an arc between nodes i and j (note: undirected case can be reduced to directed one by substituting undirected edge with two directed arcs) and so, for any time interval $[t_1, t_2]$ amount of product that can be send from i to j by corresponding arc is limited to $\int_{t_1}^{t_2} c_{ij}(t)dt$. Next, we assume that there is some mapping between the the pairs of nodes and several first natural numbers, B is the corresponding incidence matrix, if e connects i to j then $B_{ie} = 1$, $B_{je} = -1$ and $B_{ke} = 0$ for all other k. With the mentioned mapping $c_{ij}(t)$ forms the vector $\mathbf{c}(t)$ and $u_{ij}(t)$ forms the vector $\mathbf{u}(t)$. $\mathbf{x}^-$ is the initial product distribution and $\mathbf{x}^+$ is a demand vector. Thus, the problem is to transit the state from $\mathbf{x}^-$ to $\mathbf{x}^+$ in the minimum possible time.

Throughout the paper we will denote the optimal solution of (1) with input $\mathbf{x}^-, \mathbf{x}^+, \mathbf{c}$ as $\tau^*(\mathbf{x}^-, \mathbf{x}^+, \mathbf{c})$ or simply $\tau^*(\mathbf{c})$ if the other two arguments do not influence the context.

If all the $c_{ij}(t)$ does not vary over time then (1) becomes rather simple optimization problem – a linear programming problem and *parametric flow problem* in particular (see Malkovskii (2015)). It can be easily explained by the fact that with constant capacities optimal control can be searched without loss in class of constant control (that is, $u_{ij}(t)$ does not vary over time as well).

3. MAIN RESULTS

First, one simple lemma is required to proceed for the main result.

Lemma 1. Consider (1) with input $\mathbf{x}^-, \mathbf{x}^+$ and constant vector-function $\mathbf{c}(t) \equiv \bar{\mathbf{c}}$. There exists constant optimal control $\mathbf{u}(t) \equiv \bar{\mathbf{u}}$ such that subgraph $G' = < V, E' >$ with $E' = \{(i,j) \in E \mid \bar{u}_{ij} > 0\}$ does not contain a cycle.

Proof. *Existence of constant control*: let $\mathbf{u}^*(t)$ be some optimal control. Then let

$$\bar{\mathbf{u}} \equiv \frac{1}{\tau^*}\int_0^{\tau^*} \mathbf{u}^*(t)dt.$$

Due to this definition

$$\int_0^{\tau^*} \bar{\mathbf{u}}(t)dt = \int_0^{\tau^*} \mathbf{u}^*(t)dt,$$

thus $\bar{\mathbf{u}}$ transits the state from $\mathbf{x}^-$ to $\mathbf{x}^+$ in time τ^*. Due to linearity of trajectory induced by $\bar{\mathbf{u}}$ and the fact that $\mathbf{x}^- \geq 0$ and $\mathbf{x}^+ \geq 0$, $\bar{\mathbf{u}}$ produces valid trajectory for and thus, $\bar{\mathbf{u}}$ is an optimal control as well.

Existence of acyclic control: Let $\bar{\mathbf{u}}$ be such constant optimal control that minimizes quadratic form

$$\sum_{i,j=1}^{n} \bar{u}_{ij}^2.$$

Such control cannot contain a cycle: if there is a cycle, by reducing u_{ij} on that cycle by the same value the trajectory will not change (due to the fact that cycles form a kernel of incidence matrix B) and mentioned quadratic form will have decreased value and thus we have a contradiction.

Next, lets describe the class of capacity function which we are considering:

A1. We will call the function $f : [0; +\infty] \to \mathbb{R}$ *averagable* if the next limit exists

$$0 < avg(f) = \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^{\tau} f(t)dt < +\infty.$$

Next, we need two auxiliary lemmas about averagable functions:

Lemma 2. If f is averagable function then for any $T > 0$ and $0 < \varepsilon < 1$ there exists $-\infty < \sigma \leq 0$ such that

$$\forall\tau \geq 0 \quad \int_T^{T+\tau} f(t)dt \geq avg(f)(1-\varepsilon)(\tau+\sigma).$$

Proof. Due to $\frac{1}{\tau}\int_T^{T+\tau} f \to avg(f)$ for any $\varepsilon > 0$ exists τ' such that

$$\forall\tau > \tau' : \ \frac{1}{\tau}\int_T^{T+\tau} f(t)dt > avg(f)(1-\varepsilon)$$

and so if $\sigma \leq 0$

$$\forall\tau > \tau' : \ \int_T^{T+\tau} f(t)dt > avg(f)(1-\varepsilon)(\tau+\sigma).$$

Thus we need to simply take σ as

$$\sigma := \min_{\tau\in[0,\tau']} \frac{1}{avg(f)(1-\varepsilon)}\int_T^{T+\tau} f(t)dt - \tau.$$

Since for $\tau = 0$ expression under the min takes the value of 0 then we have that $\sigma \leq 0$ and thus satisfying the inequality for $\tau > \tau'$.

Lemma 3. If f is averagable function then for any $T > 0$ and $\varepsilon > 0$ there exists $0 \geq \sigma < +\infty$ such that

$$\forall\tau \geq 0 \quad \int_{T+\sigma}^{T+\sigma+\tau} f(t)dt \leq avg(f)(1+\varepsilon)(\tau+\sigma).$$

Proof. Due to $\frac{1}{\tau}\int_T^{T+\tau} f \to avg(f)$ for any $\varepsilon > 0$ exists τ' such that

$$\forall\tau \geq \tau' : \ \int_T^{T+\tau} f(t)dt < avg(f)(1+\varepsilon)\tau.$$

Thus, taking $\sigma = \tau'$ we get that

$$\begin{aligned} \forall\tau \geq 0 \quad \int_{T+\sigma}^{T+\sigma+\tau} f(t)dt &\leq \int_T^{T+\sigma+\tau} f(t)dt \\ &\leq avg(f)(1+\varepsilon)(\tau+\sigma). \end{aligned}$$

Finally, next theorem describes how to get a suboptimal solution of (1) with averagable capacity functions.

Theorem 1. Consider (1) with averagable capacity functions $c_{ij}(t)$. Let $\{\mathbf{x}_k^-\}_{k=1}^{\infty}$ and $\{\mathbf{x}_k^+\}_{k=1}^{\infty}$ be such sequences that

$$\lim_{k\to\infty} \tau^*(\mathbf{x}_k^-, \mathbf{x}_k^+, \mathbf{c}) = +\infty$$