Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-1 (2016) 059–064

Optimization Based Constrained Gaussian Sum Unscented Kalman Filter

Krishna Kumar Kottakki, Mani Bhushan and Sharad Bhartiya

Indian Institute of Technology Bombay, Mumbai, India 400076

Abstract: This work presents a novel constrained nonlinear state estimation approach for nonlinear dynamical systems. The proposed approach combines two key elements from well know Gaussian Sum Unscented Kalman Filter (GS-UKF) and Unscented Recursive Nonlinear Dynamic Data Reconciliation (URNDDR) approaches. The proposed approach uses sum of Gaussians representation in GS-UKF and explicit constrained update in URNDDR to obtain feasible state estimates. The benefits of the proposed approach are demonstrated over the available constrained GS-UKF variants using a three state isothermal batch process case study available in literature.

Keywords: Sum of Gaussians, nonlinear state estimation, interval constraints, constrained update, Bayes' rule, Kalman Filter.

1. INTRODUCTION

Recursive Bayesian Estimation algorithms are widely used in control, optimization and process monitoring applications. Bayesian estimation algorithms use available nonlinear process and measurement models along with plant measurements to obtain conditional densities of states. However, a typical estimation algorithm often encounter two practical challenges:
(i) representation of non-Gaussian densities driven by nonlinear process models, and
(ii) infeasible state estimates resulted due to unconstrained Bayesian update.

Bayes' rule has closed form Kalman Filter (KF) solution [Kalman 1960] when the densities, namely, prior and likelihood are Gaussian. However, the KF solution is suboptimal if either of these densities are non-Gaussian. Further, KF solution or Bayes' rule does not account for state constraints. To overcome these limitations associated with KF, various nonlinear state estimation algorithms have been developed in literature. Extended Kalman Filter (EKF) [Anderson and Moore 1979], Unscented Kalman Filter (UKF), Julier and Uhlmann [2004], Ensemble Kalman Filter (EnKF) [Gillijns et al. 2006], Particle Filter (PF) [Arulampalam et al. 2002] and Gaussians Sum Filters (GS-F) [Sorenson and Alspach 1971, Kotecha and Djuric 2003, Šimandl 2005] are the well known nonlinear state estimation algorithms available in literature.

Among the estimation algorithms, EKF is limited to systems with near Gaussian densities since it linearizes process and measurement models to transform the moments of state conditional densities [Julier and Uhlmann 2004]. Whereas the other approaches, namely, UKF, EnKF and PF use set of samples to represent non-Gaussian densities and does not require to perform linearization step as in EKF. Instead, samples will be used to transform conditional densities. Among these three sampling based approaches, UKF has gained much attention [Qu and Hahn 2009, Julier and Uhlmann 2004, Kandepu et al. 2008] since it uses only deterministic choice of $2n+1$ samples, known as sigma points. Here n is dimension of process states. Where as, EnKF and PF use Monte Carlo based sampling, which in principle require large number of samples [Arulampalam et al. 2002]. UKF assumes that the prior density to be Gaussian, a common violation for nonlinear/non-Gaussian systems and does not address the objective of representation of nonlinear/non-Gaussian [Arulampalam et al. 2002]. Gaussian Sum Unscented Kalman Filter (GS-UKF), belongs to the class GS-F, overcomes the limitations of UKF by representing non-Gaussian densities with sum of Gaussians. GS-F are developed with a premise that "sum of Gaussians can approximate any density to an arbitrary degree of accuracy [Sorenson and Alspach 1971]. To transform sum of Gaussians through nonlinear process model, GS-UKF adopts sigma points at each Gaussian and subsequently obtain sum of Gaussian prior density [Šimandl 2005]. Further, update step using Bayes' rule results in sum of Gaussians posterior density. Other GS-F variants namely, Gaussian Sum - Extended Kalman Filter [Sorenson and Alspach 1971] and Gaussian Sum - Particle Filter [Kotecha and Djuric 2003]. However, with the choice of EKF and PF at each Gaussian of sum of Gaussians, these approaches also suffer form the issues associated with EKF and PF, respectively.

While GS-UKF has addressed the problem of nonlinear or non-Gaussian representation, unconstrained Bayesian update step can lead to infeasible state estimates. In literature, various strategies are developed to address this problem, which can be classified into two strategies.
(i) incorporates constraints on unconstrained posterior moments, using projection [Teixeira et al. 2010] or density truncation steps [Simon 2010]
(ii) modifies unconstrained update step in Bayes' rule into an explicit constrained optimization problem [Vachhani et al. 2005, 2006].

⋆ The authors thank the Department of Science and Technology, India, for partial financial assistance under grant 13DST057.

10.1016/j.ifacol.2016.03.029

In literature, Projection based Gaussian Sum Constrained Unscented Kalman Filter (PC-GS-UKF) by Ishihara and Yamakita [2009] and density Truncation based Gaussian Sum Unscented Kalman Filter (TC-GS-UKF) by Straka et al. [2012], are the available extensions of constrained GS-UKF. Both these approaches use unconstrained update step followed by constraint incorporation step. Moreover these approaches use unconstrained posterior moments in subsequent time steps. Thus, these developments have a severe limitation that the constrained state estimates are based on the quality of unconstrained posterior moments. To overcome this problem, in this work, we present a constrained GS-UKF approach which incorporates state constraints through an explicit optimization problem. Thus, the proposed approach can use constrained posterior moments in subsequent time instants.

Organization of this paper is as follows: Section 2 presents the problem statement for constrained nonlinear state estimation. Section 3 presents Unscented Recursive Nonlinear Dynamic Data Reconciliation (URNDDR) constrained UKF approach. Section 4 presents the proposed Optimization Based Constrained Gaussian Sum Unscented Kalman Filter (OC-GS-UKF) approach and its utility is demonstrated in Section 5, using a three state isothermal batch process available in literature. This paper is concluded in Section 6.

2. PROBLEM STATEMENT

Consider a sampled data system consisting of nonlinear process dynamics, a linear measurement function and interval (bound) constraints on the states as,

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))dt + \mathbf{w}(t_k), \tag{1}$$

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}(t_k) + \mathbf{v}_k, \tag{2}$$

$$\mathbf{x}(t_0) \sim \mathcal{N}(\hat{\mathbf{x}}_{0|0}, \mathbf{P}_{0|0}) \tag{3}$$

$$\mathbf{x}_L \leq \mathbf{x}(t_k) \leq \mathbf{x}_U \tag{4}$$

where, $\mathbf{x}(t_{k-1}), \mathbf{x}(t_k) \in \mathbb{R}^n$,$\mathbf{u}(t) \in \mathbb{R}^p$, represent the state and input vectors at time t while $\mathbf{y}_k \in \mathbb{R}^m$, $\mathbf{w}(t_k) \in \mathbb{R}^n$, $\mathbf{v}_k \in \mathbb{R}^m$ represent observation, state noise and measurement noise, respectively at time t_k. Further, $\mathbf{w}(t_k) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ are assumed to be independent Gaussian, white, stochastic processes The initial state is unknown and assumed to have a Gaussian distribution as in Eq. (3). Function $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ represents the nonlinear state dynamics and $\mathbf{H} \in \mathbb{R}^{m\times n}$ represents the linear observation model. Measurements $\mathbf{y}_k$ are assumed to be available at regularly spaced sampling instants t_k, at $k = 0, 1, 2, 3, \ldots$ with $T_s = t_k - t_{k-1}$ being the sampling period. For ease of notation, we define $\mathbf{x}_k \triangleq \mathbf{x}(t_k)$. Eq. (4) specifies the interval (bound) constraints on each component of state vector $\mathbf{x}_k$. The filtering problem is to find a feasible point estimate for $\mathbf{x}_k$, governed by dynamics in Eq. (1), using available measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ which are related to the states as in Eq. (2) and subjected to constraints given in Eq. (4). In Eq. (4), $\mathbf{x}_L$ and $\mathbf{x}_U$ represents the lower and upper bounds on states, respectively.

Now we will present the proposed constrained UKF approach which represents underlying densities with single Gaussian and uses an explicit optimization problem to incorporate state constraints.

3. UNSCENTED RECURSIVE NONLINEAR DYNAMIC DATA RECONCILIATION (URNDDR)

Unscented Recursive Nonlinear Dynamic Data Reconciliation approach uses a three step framework for obtaining constrained posterior moments.
(i) at $(k-1)^{th}$ time instant, it uses interval constrained unscented transformation (ICUT) approach [Vachhani et al. 2006] to generate feasible sigma points,
(ii) transforms the constrained sigma points through the process model to obtain constrained predicted sigma points at k^{th} time instant and subsequently represents the prior with a Gaussian density, and
(iii) performs constrained update step to obtain feasible posterior moments at k^{th} time instant.

The details of these three steps are presented next.

3.1 Constrained sigma point generation using interval constrained unscented transformation

Consider a Gaussian random variable $\mathbf{x}_{k-1}$ with mean $\hat{\mathbf{x}}_{k-1|k-1}$ and covariance $\mathbf{P}_{k-1|k-1}$ at time t_{k-1}. The ICUT approach deterministically selects $2n+1$ constrained sigma points [Vachhani et al. 2006, Teixeira et al. 2010],

$$\boldsymbol{\chi}_{k-1|k-1}^{(i,c)} = \begin{cases} \hat{\mathbf{x}}_{k-1|k-1}, & i = 0 \\ \hat{\mathbf{x}}_{k-1|k-1} + \theta_i[\sqrt{\mathbf{P}}]_i, & i = 1, 2, \ldots, n \\ \hat{\mathbf{x}}_{k-1|k-1} - \theta_{i-n}[\sqrt{\mathbf{P}}]_{i-n}, & i = n+1, n+2, \ldots, 2n \end{cases} \tag{5}$$

and corresponding weights as,

$$w_{k-1}^{(i,c)} = \begin{cases} b_{k-1}, & i = 0 \\ a_{k-1}\theta_i + b_{k-1}, & i = 1, 2, \ldots, n \\ a_{k-1}\theta_{i-n} + b_{k-1}, & i = n+1, n+2, \ldots, 2n \end{cases} \tag{6}$$

where

$$a_{k-1} = \frac{(2\kappa - 1)}{2(n+\kappa)\left[\sum_{i=1}^{2n} \theta_{i,k-1} - (2n+1)\sqrt{n+\kappa}\right]} \tag{7}$$

$$b_{k-1} = \frac{1}{2(n+\kappa)} - a_{k-1}\sqrt{n+\kappa} \tag{8}$$

$$\theta_{i,k-1} = \min_{1\leq j\leq n} (\Theta_{j,i}), \quad i = 1, 2, \ldots, 2n \tag{9}$$

$$\text{where, for } \; j = 1, 2, \ldots, n; \; i = 1, 2, \ldots, 2n$$

$$\Theta_{j,i} \triangleq \begin{cases} \sqrt{n+\kappa} & \text{if } \mathbf{S}_{j,i} = 0 \\ \min\left(\sqrt{n+\kappa}, \dfrac{\mathbf{x}_{L_j} - \hat{\mathbf{x}}_{j,k-1|k-1}}{\mathbf{S}_{(j,i)}}\right) & \text{if } \mathbf{S}_{j,i} < 0 \\ \min\left(\sqrt{n+\kappa}, \dfrac{\mathbf{x}_{U_j} - \hat{\mathbf{x}}_{j,k-1|k-1}}{\mathbf{S}_{(j,i)}}\right) & \text{if } \mathbf{S}_{j,i} > 0 \end{cases} \tag{10}$$

$$\text{with } \; \mathbf{S} \triangleq \left[\sqrt{\mathbf{P}_{k-1|k-1}} \; -\sqrt{\mathbf{P}_{k-1|k-1}}\right] \tag{11}$$

In Eq. (5), $\boldsymbol{\chi}_{k-1|k-1}^{(i,c)}, w_{k-1}^{(i,c)}$ represent the i^{th} constrained sigma point and its corresponding weights, respectively. In absence of state constraints, the constrained sigma point selection procedure will converges to well known unscented transformation approach [Julier and Uhlmann 2004].

3.2 Propagation of sigma points:

The constrained sigma points $\boldsymbol{\chi}_{k-1|k-1}^{(i,c)}$ obtained in Eq. (5) are then propagated through the process model (Eq. (1)) to obtain the predicted sigma points $\boldsymbol{\chi}_{k|k-1}^{(i,c)}$ as,