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Monte Carlo Gaussian Sum Filter For State Estimation of Nonlinear Dynamical Systems

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Abstract: This work presents a novel nonlinear/non-Gaussian state estimation algorithm, named as, Monte Carlo Gaussian Sum Filter (MC-GSF). The proposed approach combines the elements of Monte Carlo (MC) sampling and design choices in recently developed Unscented Gaussian Sum Filter (UGSF). While the MC sampling retains the sampling benefits in capturing moments of non-Gaussian densities, the design choices in UGSF improves the ability of MC samples by means of sum of Gaussians representation. Further, the design choices in UGSF also overcomes the potential degeneracy issues persisting with Particle filters and Gaussian Sum Filters. We demonstrate the superiority of proposed approach by implementing on an illustrative case study.

Keywords: sum of Gaussians, nonlinear state estimation, particle filter, unscented Gaussian sum filter

1. INTRODUCTION

Nonlinear Recursive Bayesian estimation algorithms are widely used in control [Downs and Vogel 1993, Lee and Ricker 1994], optimization, process monitoring and parameter estimation [Evensen 2009]. These algorithms use a two step approach to obtain state estimates, namely, (i) transforms available conditional densities through nonlinear process model to obtain model predictions, and (ii) updates the current model predictions with available measurement(s). However, these two steps are further involved in various challenges for online implementation and are as follows [Patwardhan et al. 2012],
1. the conditional densities of states need to be transformed through nonlinear process model in a limited computational efforts,
2. accurate representation of resulting an unknown non-Gaussian but arbitrary densities, and
3. state estimates must be feasible.

In literature, Extended Kalman Filter (EKF) [Anderson and Moore 1979], Ensemble Kalman Filter (EnKF) [Gillijns et al. 2006], Unscented Kalman Filter (UKF) [Julier and Uhlmann 2004], Particle Filter (PF) [Arulampalam et al. 2002], Gaussians Sum Filters (GS-F)[Sorenson and Alspach 1971] and recently developed Unscented Gaussian Sum Filter (UGSF) [Kottakki et al. 2014c] have been developed to address the challenges associated with nonlinear/non-Gaussian estimation.

Among these estimation algorithms, EKF is limited only to the systems exhibit near Gaussian densities as it has an assumption of Gaussianity in representation of underlying densities [Julier and Uhlmann 2004]. While this assumption allows EKF to make use of Kalman Filter (KF) [Kalman 1960] expressions, the update step in KF can also lead to infeasible state estimates [Simon 2010]. To overcome these issues, sampling based algorithms, namely, UKF, UGSF, EnKF and PF are developed in literature which use of a set of samples to represent non-Gaussian densities. While UKF and UGSF use Unscented Transformation (UT) based deterministic sampling approach, EnKF and PF use Monte Carlo (MC) to represent state conditional densities. UT chooses 2n+1 samples, known as sigma points, and corresponding design weights to represent densities [Julier and Uhlmann 2004]. Here, n represents the dimension of state vector. MC random sampling, where it requires significantly large number of samples.

Among these sampling based approaches, an additional key difference is in representation of underlying non-Gaussian densities using the predicted samples which in turn has lead to different update steps. EnKF updates the individual predicted samples using KF expressions and these updated samples will be used in subsequent time step(s). The usage of KF expressions has an assumption that the prior densities are Gaussian [Evensen 2009]. Thus, EnKF potentially suffers from the issues lead by KF approach. Whereas, in PF, weights of the predicated samples are updated using innovations of respective samples. The updated weights of samples along with predicted samples are used in subsequent time instants. Further, PF performs resampling step to overcome weight degeneracy issue associated with the weight update step [Arulampalam et al. 2002]. This resampling step requires an importance density, which in turn obtained from conventional state estimation algorithms, namely, EKF, UKF and EnKF. This leads to additional computational efforts as well as bias in state estimates.

UT based approaches, namely UKF and UGSF, have a key difference in representation of non-Gaussian prior. UKF

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uses only the first two moments, i.e mean and covariance, of predicted sigma points in representing the non-Gaussian prior. This has an assumption that the prior density is Gaussian and allows UKF to make use of KF expressions. This assumption not only under represent the non-Gaussian densities but also under determines the ability of sigma points in capturing nonlinear/non-Gaussian densities [Kottakki et al. 2014c]. UGSF represents the prior as a sum of Gaussians by assigning predicted sigma points as means of Gaussians. Further, it uses the weights of sigma points in UT as the weights of each Gaussian and uncertainty of states as covariance to each Gaussian to obtain sum of Gaussians representation [Kottakki et al. 2014c]. These design choices results in posterior moments of UGSF as a sum of Gaussians. Further, to overcome the degeneracy issue associated with the weight update step in UGSF, it reapproximates the posterior with a single Gaussian. Note that, this reapproximation step does not effect the objective of nonlinear state estimation as it requires an accurate representation of non-Gaussian prior density. The design choices in UGSF have been made to improve the ability of sigma points and retain the computational efforts equivalent to UKF. However, UGSF does not carried out apriori analysis on the design choices of sum of Gaussians representation, as it was mentioned by Sorenson and Alspach [1971]. Thus, for a typical scenarios these choices can be inadequate and can lead to poor representation of underlying non-Gaussian densities and subsequently on poor state estimates [Kottakki et al. 2014c]. To overcome these issues in this work we propose an estimation algorithm named as, Monte Carlo Gaussian Sum Filter (MC-GSF) which uses Monte Carlo samples with a premise that large number of samples can capture the nonlinear/non-Gaussian densities. Further, it combines the design choices in UGSF, to represent non-Gaussian densities as a sum of Gaussians and to overcome the degeneracy issues.

The organization of this paper is as follows: Section 2 presents the problem statement for nonlinear state estimation. Section 3 presents recently developed Sum of Gaussians based Unscented Gaussian Sum Filter (UGSF) and Section 4 presents the proposed Monte Carlo Gaussian sum filter (MC-GSF). The utility of proposed MC-GSF is illustrated using a benchmark case study is presented in Section 5. This paper is concluded in Section 6.

2. PROBLEM STATEMENT

Consider a sampled data system consisting of nonlinear process dynamics, a linear measurement function and interval (bound) constraints on the states as,

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))dt + \mathbf{w}(t_k), \tag{1}$$

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}(t_k) + \mathbf{v}_k, \tag{2}$$

$$\mathbf{x}(t_0) \sim \mathcal{N}(\hat{\mathbf{x}}_{0|0}, \mathbf{P}_{0|0}) \tag{3}$$

$$\mathbf{x}_{LB} \leq \mathbf{x}(t_k) \leq \mathbf{x}_{UB} \tag{4}$$

In Eq. (1), $\mathbf{x}(t_k)$, $\mathbf{x}(t_{k-1}) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^p$, $\mathbf{w}(t_k) \in \mathbb{R}^n$ represent the state, input and state noise vectors, respectively. In Eq. (2), $\mathbf{y}_k \in \mathbb{R}^m$, $\mathbf{v}_k \in \mathbb{R}^m$ represent observation, and measurement noise vectors, respectively. Further, $\mathbf{w}(t_k) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ are assumed to be independent Gaussian, white, stochastic processes. Initial state at time t_0 is assumed to be have a Gaussian distribution with mean $\mathbf{x}(t_0)$ and covariance $\mathbf{P}_{0|0}$ as given in Eq. (3). Further, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ in Eq. (1) and $\mathbf{H} \in \mathbb{R}^{m\times n}$ in Eq. (2) represent the nonlinear state dynamics and measurement models, respectively. Measurements $\mathbf{y}_k$ are assumed to be available at regularly spaced sampling instants t_k, at $k = 0, 1, 2, 3, \ldots$ with $T_s = t_k - t_{k-1}$ being the sampling period. For ease of notation, we define $\mathbf{x}_k \triangleq \mathbf{x}(t_k)$. Eq. (4) specifies the interval (bound) constraints on each component of state vector $\mathbf{x}_k$. The objective is to use Bayes' rule to find conditional densities of state estimate $\mathbf{x}_k$, governed by dynamics in Eq. (1), using available measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ which are related to the states as in Eq. (2) and subjected to interval constraints given by Eq. (4). In Eq. (4), $\mathbf{x}_{LB}$ and $\mathbf{x}_{UB}$ represents the lower and upper bounds on states, respectively. Details and challenges in using Bayes' rule for obtaining presented next.

2.1 Bayesian Estimation:

Bayes' rule for obtaining the posterior density is given as [Maybeck 1979],

$$p_{\mathbf{x}_k|\mathbf{Y}_k}(\boldsymbol{\xi}_k|\mathbf{Y}_k) = \frac{p_{\mathbf{x}_k|\mathbf{Y}_{k-1}}(\boldsymbol{\xi}_k|\mathbf{Y}_{k-1}) p_{\mathbf{y}_k|\mathbf{x}_k,\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\boldsymbol{\xi}_k, \mathbf{Y}_{k-1})}{p_{\mathbf{y}_k|\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\mathbf{Y}_{k-1})} \tag{5}$$

In Eq. (5), $p_{\mathbf{x}_k|\mathbf{Y}_k}(\boldsymbol{\xi}_k|\mathbf{Y}_k)$ represents the conditional posterior density of state $\mathbf{x}_k$. $p_{\mathbf{x}_k|\mathbf{Y}_{k-1}}(\boldsymbol{\xi}_k|\mathbf{Y}_{k-1})$ is the prior density evolved after transforming initial conditional density of states (Eq. (3)) at $(k-1)^{th}$ time instant through the process model (Eq. (1)), $p_{\mathbf{y}_k|\mathbf{x}_k,\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\boldsymbol{\xi}_k, \mathbf{Y}_{k-1})$ is the likelihood density associated with current measurement $\mathbf{y}_k$ (Eq. (2)) and is given as [Maybeck 1979],

$$p_{\mathbf{y}_k|\mathbf{x}_k,\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\boldsymbol{\xi}_k, \mathbf{Y}_{k-1}) = \frac{1}{(2\pi)^{m/2}|\mathbf{R}|^{1/2}} \exp\{\cdot\}_2 \tag{6}$$

$$\{\cdot\}_2 = \frac{-1}{2}[\boldsymbol{\zeta}_k - \mathbf{H}(\boldsymbol{\xi}_k)]^T \mathbf{R}^{-1}[\boldsymbol{\zeta}_k - \mathbf{H}(\boldsymbol{\xi}_k)] \tag{7}$$

In Eq. (5), denominator term is known as evidence or constant of integration associated with prior and likelihood densities and is given as [Maybeck 1979],

$$p_{\mathbf{y}_k|\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\mathbf{Y}_{k-1}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{\mathbf{x}_k|\mathbf{Y}_{k-1}}(\boldsymbol{\xi}_k|\mathbf{Y}_{k-1}) \, p_{\mathbf{y}_k|\mathbf{x}_k,\mathbf{Y}_{k-1}}(\boldsymbol{\zeta}_k|\boldsymbol{\xi}_k, \mathbf{Y}_{k-1}) d\boldsymbol{\xi}_k \tag{8}$$

Among these (Eqs. (5) to (8)), representing prior density and performing multivariate integration (Eq.(8)) are the challenging in any nonlinear Bayesian estimation algorithm. Now we will present the UGSF approach which addresses these challenges by using sum of Gaussians.

3. UNSCENTED GAUSSIAN SUM FILTER

Unscented Gaussian sum filter, uses UT to represent non-Gaussian densities using sigma points. These are subsequently transformed through the process model to obtain predicted sigma points, which in turn will be used represent prior density as a sum of Gaussians. The details are presented next [Kottakki et al. 2014c]: