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Mean-field stochastic Volterra optimal singular control with Poisson jumps ⋆

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Abstract: This paper considers an infinite horizon mean-field type of stochastic Volterra singular control problem. The dynamical system is governed by a Itô-Lévy processes and a standard one dimensional independent Brownian motion. Stochastic Volterra controlled system is difficult to manipulate by standard methods such as dynamic programming and classical maximum principle, because the presence of memory terms in the dynamics of the system. In this study, Malliavin calculus is a useful tool to overcome these difficulties. Necessary and sufficient condition for the aforementioned system is established by using Malliavin calculus, convex perturbation technique and transversality condition. Finally an example is given to show that the application of the theoretical study.

Keywords: Malliavin calculus, Mean-field theory, Optimal control, Singular control, Stochastic system, Volterra equation.

1. INTRODUCTION

Mean-field stochastic optimal control problems have received a considerable research attention in the recent years due to the wide applicability of several different areas such as biology, game theory, economics and finance, see Bensoussan et al. (2013). In contrast to standard stochastic control problems both the dynamical system and the performance functional involves mean of state variables. An example for this type of problem is continuous time Markowitz's mean-variance portfolio selection model where the variance term involves quadratic function of expectation see Zhou et al. (2000).

Stochastic singular control problem has an active area of research and an evident applications. Singular control problems exhibit many interesting, deep theoretical niceties and has practical significance, in engineering fields other than aerospace see Naidu et al. (2001). Also has applications in non-engineering areas such as economics, see Hafayed (2014). This motivations take much effort which has been put into the development of a new theory that deals with stochastic singular control problems. In this type of problems, the control variable has two components, first is being absolutely continuous and second is singular.

Stochastic Volterra integral equations arise in many scientific applications such as the Volterra population growth model, biological species living together, propagation of stocked fish in new lake, heat transfer and heat radiation are described by integral equations see Corduneanu et al. (2000). Stochastic Volterra equations are not Markov processes and therefore classical methods such as dynamic programming cannot be used for such equations. However, we show that using Malliavin calculus is possible to formulate modified functional type of maximum principle for such systems, see Nunno et al. (2000); Øksendal (2010); Agram et al. (2014).

From the above applications it stimulates to study the mean-field type optimal singular control where the system evolves stochastic Volterra equation governed by Poisson jump processes and independent Brownian motion. Hamiltonian and adjoint processes also have Malliavin derivative which made the computations easy.

The rest of this paper is structured as follows: Section 2 begins with the basic concepts of Malliavin calculus for Lévy processes and Brownian motion. Section 3 and section 4 are devoted to prove the main result which is necessary and sufficient condition of the prescribed system. A suitable example is given to exhibit the main results in section 5.

2. PRELIMINARIES AND FORMULATION OF THE STOCHASTIC SINGULAR CONTROL PROBLEM

In this section, we recollect some basic definitions, properties of Malliavin calculus for Lévy processes related to the discussion, and to formulate the problem.

2.1 Lévy processes and Brownian motion

A real valued stochastic processes $\mathfrak{L} = \{\mathfrak{L}(t), t \geq 0\}$ is defined in a complete probability space $\left(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ is called Lévy processes, if $\mathfrak{L}$ has a stationary and independent increments with $\mathfrak{L}(0) = 0$, and $\mathfrak{L}(t)$ is continuous

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in probability, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual condition. i.e., the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is a right continuous increasing family of complete sub σ- algebra of $\mathcal{F}$.
Let $W(\cdot) = W(t)_{t\geq 0}$ be a standard one dimensional Brownian motion, and

$$\chi(\cdot) := \int_0^t \int_{\mathbb{R}_0(:=\mathbb{R}-\{0\})} a\tilde{N}(ds, da),$$

is an independent pure jump Lévy martingale on a given filtered probability space. Here $a \in \mathbb{R}_0$ and $\tilde{N}(dt, da) = N(dt, da) - \nu(da)dt$ is the compensated jump measure on $\chi(\cdot)$, where $N(dt, da)$ is the jump measure and $\nu(da)$ is the Lévy measure of the Lévy processes $\chi(\cdot)$.
Let $\mathcal{A}_1$ be a closed convex subset of $\mathbb{R}$ and $\mathcal{A}_2 := [0, \infty)$. Let $\mathcal{U}_1$ be the class of measurable, $\mathcal{F}_t$- adapted processes $u(\cdot) : [0, \infty) \times \Omega \to \mathcal{A}_1$, and $\mathcal{U}_2$ is the class of measurable, $\mathcal{F}_t$- adapted processes $\eta(\cdot) : [0, \infty) \times \Omega \to \mathcal{A}_2$. The definition of the singular part of the admissible control is follows:

Definition 1. An admissible control is a pair $(u(\cdot), \eta(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2$- valued, $\mathcal{F}_t$- adapted processes, such that

(i) $\eta(\cdot)$ is a bounded variation, non-decreasing continuous on the left with right limits and $\eta(0) = 0$,
(ii) $E[\sup_{t\in[0,T]} |u(t)|^2 + |\eta(t)|^2] < \infty$.

Here $\mathcal{U}_1 \times \mathcal{U}_2$ be the set of all admissible controls. We note that since $d\eta(t)$ may be singular with respect to Lebesgue measure dt, we call $\eta(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part.

2.2 Malliavin calculus for Lévy processes

In view of the Lévy-Itô decomposition theorem, which states that any Lévy processes $\mathfrak{L}(t)$ with

$$E\left[\mathfrak{L}^2(t)\right] < \infty \quad \text{for all} \quad t,$$

can be written

$$\mathfrak{L}(t) = at + bW(t) + \int_0^t \int_{\mathbb{R}_0} a\tilde{N}(ds, da),$$

with constants a and b, we see that it suffices to deal Malliavin calculus for $W(\cdot)$ and for $\chi(\cdot)$ separately.

2.3 Malliavin calculus for Brownian motion

A natural starting points is the Wiener-Itô chaos expansion theorem, which states that any $F \in L^2(\mathcal{F}_T, P)$ can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{1}$$

for a unique sequence of symmetric deterministic functions $f_n \in L^2(\lambda^n)$, where λ is Lebesgue measure on $[0, T]$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) \times dW(t_1)dW(t_2)\cdots dW(t_n),$$

(the n times iterated integral of f_n with respect to $W(\cdot)$) for $n = 1, 2, \ldots$ and $I_0(f_0) = f_0$ when f_0 is a constant. Moreover, we have the isometry

$$E\left[F^2\right] = ||F||^2_{L^2(p)} = \sum_{n=0}^{\infty} n!||f_n||^2_{L^2(\lambda^n)}.$$

Definition 2. (Malliavin derivative D_t with respect to $W(\cdot)$) Let $\mathbb{D}^{(W)}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos exapansion (1) satisfies

$$||F||^2_{\mathbb{D}^{(W)}_{1,2}} := \sum_{n=1}^{\infty} nn!||f_n||^2_{L(\lambda^n)^2} < \infty.$$

For $F \in \mathbb{D}^{(W)}_{1,2}$ and $t \in [0, T]$, we define the Malliavin derivative of F at t(with respect to $B(\cdot)$), $D_t F$, by

$$D_t F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)),$$

where the notation $I_{n-1}(f_n(\cdot, t))$ means that we apply the $(n-1)$ times iterated integral to the first $n-1$ variables $t_1, \ldots, t_{n-1}$ of $f_n(t_1, t_2, \ldots, t_n)$ and keep the last variable $t_n = t$ as a parameter.
One can easily check that

$$E\left[\int_0^T (D_t F)^2 dt\right] = \sum_{n=1}^{\infty} nn!||f_n||^2_{L(\lambda^n)^2} = ||F||^2_{\mathbb{D}^{(W)}_{1,2}}.$$

so $(t, \beta) \to D_t F(\beta)$ belongs to $L^2(\lambda \times P)$.

Lemma 3. (The generalized duality formula for $W(\cdot)$). Let $F \in L^2(\mathcal{F}_T, P)$ and let $\varphi(t, \beta) \in L^2(\lambda \times P)$ be adapted. Then,

$$E\left[F \int_0^T \varphi(t)dW(t)\right] = E\left[\int_0^T E[D_t F|\mathcal{F}_t]\varphi(t)dt\right].$$

2.4 Malliavin calculus for Poisson jump

The construction of a stochastic derivative or Malliavin derivative in the pure jump martingale case follows the same lines as in the Brownian motion case. In this case, the corresponding Wiener-Itô chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$(where in this case, $\mathcal{F}_t = \mathcal{F}^{(\tilde{N})}_t$ is the σ-algebra generated by $\chi(\cdot); 0 \leq s \leq t$) can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n); f_n \in \hat{L}^2((\lambda \times \nu)^n), \tag{2}$$

where $\hat{L}^2((\lambda \times \nu)^n)$ in the space of functions $f_n(t_1, a_1, \ldots, t_n, a_n)$, $t_i \in [0, T], a_i \in \mathbb{R}_0$ such that $f_n \in \hat{L}^2((\lambda \times \nu)^n)$ and f_n is symmetric with respect to the pairs of variables $(t_1, a_1), \ldots, (t_n, a_n)$.
It is important to note that in this case, the n times iterated integral $I_n(f_n)$ is taken with respect to $\tilde{N}(dt, da)$ and not with respect to $d\chi(t)$. Thus, we define

$$I_n(f_n) = n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \cdots \int_0^{t_2} \times \int_{\mathbb{R}_0} f_n(t_1, a_1, \ldots, t_n, a_n)\tilde{N}(dt_1, da_1)\cdots\tilde{N}(dt_n, da_n),$$

for $f_n \in \hat{L}^2((\lambda \times \nu)^n)$.
The Itô isometry for stochastic integrals with respect to $\tilde{N}(dt, da)$ then gives the following isometry for the chaos expansion:

$$||F||^2_{L^2(P)} = \sum_{n=0}^{\infty} n!||f_n||^2_{L^2((\lambda\times\nu)^n)}.$$