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Controllability study on fractional order impulsive stochastic differential equation ⋆

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Abstract: In this paper, we consider a fractional order stochastic differential system with impulsive conditions. The necessary and sufficient conditions for the controllability of associated linear stochastic system is studied by using the controllability Grammian matrix defined by Mittag-Leffler function. The sufficient condition for controllability of the proposed nonlinear system is proved by using Banach fixed point theorem. An example is provided to illustrate the theory using the numerical integration by Haar wavelet approximation method.

Keywords: Complete Controllability; Stochastic system; Impulsive fractional differential system; Mittag-Leffler function; Laplace transform; Numerical integration.

1. INTRODUCTION

Many dynamic processes are characterized by the fact that at certain moments of time they experience sudden changes of state. These changes may seem instantaneous because the durations of these changes are negligible in comparison with the duration of the whole process. Therefore, it is natural to assume that these changes are in the form of impulses. Dynamic systems subject to impulsive effects are defined as impulsive systems. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, do exhibit impulsive effects, see Lakmikantham (1998), and Karthikeyan (2011).

An increasing interest in issues related to fractional dynamical systems oriented towards the field of control theory can be seen from the literature, for instance, see Nadeem (2014). Stochastic differential equations have many applications in economics, ecology and finance. In recent years, the controllability problems for stochastic differential equations have become a field of increasing interest, (see Karthikeyan (2009) and references therein). The extensions of deterministic controllability concepts to stochastic control fractional systems have been discussed only in a limited number of publications, see Nadeem (2014). The question of why would we need impulsive control may arise. In some cases, impulsive controls are preferred over continuous controls.The important consideration is that impulsive control could be more practical and cheaper than continuous control. For example, in spacecraft formation control problems, see Lakmikantham (1998).

In this paper, we consider the controllability of nonlinear fractional stochastic differential systems with impulses as follows:

$$\begin{aligned} &{}^C D^\alpha x(t) = Ax(t) + Gu(t) + f(t, x(t)) \\ &\qquad +\sigma(t, x(t))\frac{dw(t)}{dt}, t = [t_0, T] \setminus \{t_1, t_2, \ldots, t_\rho\} \\ &\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), k = 1, 2, \ldots, \rho, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where ${}^C D^\alpha x(t)$ denotes an α order Caputo's fractional derivative of $x(t), 0 < \alpha < 1$, A and G are the known constant matrices and satisfy $A \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input. $w(t)$ is a given $l-$ dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s), 0 \le s \le t$ and $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are appropriate continuous functions. $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for $k = 1, 2, \ldots, \rho$, and

$$x(t_k^+) = \lim_{h\to 0^+} x(t_k + h), \quad x(t_k^-) = \lim_{h\to 0^-} x(t_k - h)$$

represent the right and left limits of $x(t)$ at $t = t_k$ and the discontinuous points

$$t_0 < t_1 < t_2 < \cdots < t_\rho < t_{\rho+1} = T$$

and $x(t_k) = x(t_k^-)$ which implies that the solution of system (1) is left continuous at t_k.

This article is organized as follows: In section 2, provides some preliminaries of fractional calculus, Laplace transform and the solution of the linear fractional stochastic system. Section 3 gives the existence and uniqueness for the solution of the linear and nonlinear impulsive fractional stochastic system. Finally we provide two examples to demonstrate the effectiveness of our method.

2. PRELIMINARIES

In this section, we first recall some basic definitions of fractional calculus, which are useful for this work. Through out this paper, let Banach space, $PC([t_0, T], \mathbb{R}^n) = \{x :$

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$[t_0, T] \to \mathbb{R}^n | x \in C((t_k, t_{k+1}]), k = 0, 1, \ldots, \rho,\}$ and there exist $x(t_k^-)$ and $x(t_k^+)$, for $k = 1, 2, \ldots, \rho$, with $x(t_k^-) = x(t_k)$ with norm $\|x\|_{PC} = \sup\{\|x(t)\| : t \in J\}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathcal{P}-$null sets). Let $\alpha > 0$, with $n-1 < \alpha < n$. Let $\mathbb{R}^m$ be the $m-$dimensional Euclidean space. Let $\mathcal{C}$ denote the Banach space $PC([t_0, T], L_2(\Omega, \mathcal{F}, \mathcal{P}))$ (see Karthikeyan (2009), Guendouzi (2013) for more details.)

Definition 1. (Caputo fractional derivative) Let $f \in C[t_0, \infty)$. For $t \in [t_0, \infty)$, the Caputo fractional derivative ${}^C D^\alpha f(t)$ of order α is defined by

$$ {}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \times \int_{t_0}^{t} (t-s)^{n-\alpha-1} \left[\frac{d^n}{ds^n} f(s) \right] ds, $$

where n is positive integer such that $n - 1 \leq \alpha \leq n$. Particularly, when $0 < \alpha < 1$, it holds

$$ {}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds. $$

Definition 2. (Mittag-Leffler function) For $z \in C$, the two-parameter Mittage Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0). $$

For example,

$$ E_{\alpha,1}(z) = E_\alpha(z), \quad \textit{here } \beta = 1, $$

$$ \int_0^t E_\alpha(z^\alpha) dz = t E_{\alpha,2}(t^\alpha), $$

Laplace transform of Mittag-Leffler function is

$$ L[t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha); s] = \int_0^\infty e^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha) dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad (Re(s) > |a|^{\frac{1}{2}}), \tag{2} $$

where $Re(s)$ denotes the real parts of s. In addition, Laplace transform of $t^{\alpha-1}$ is

$$ L[t^{\alpha-1}; s] = \Gamma(\alpha) s^{-\alpha}, \quad \alpha > 0. $$

2.1 Linear fractional stochastic system

Let us consider the linear fractional stochastic differential equation of the form

$$ \begin{aligned} {}^C D^\alpha x(t) = {} & Ax(t) + \sigma(t) \frac{dw(t)}{dt} + f(t), t \in [t_0, T] \\ x(t_0) = {} & x_0, \end{aligned} \tag{3} $$

where $0 < \alpha \leq 1$, A is $n \times n$ matrix, $\sigma : [t_0, T] \to \mathbb{R}^{n\times l}$ is appropriate function and $f : [t_0, T] \to \mathbb{R}^n$ is continuous function.

Lemma 3. The solution $x(t)$ of the system (3) can be represented as

$$ x(t) = x(t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \times \left[Ax(t_0) + \left(\int_0^\eta \sigma(\theta) dw(\theta) \right) + f(s) \right] ds, t \in [0, T]. $$

Proof. Applying the idea used in [Zhou (2013)], we have the integral equation of the system (3),

$$ x(t) = x(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \times \left[Ax(s) + \sigma(s) \frac{dw(s)}{ds} + f(s) \right] ds, $$

since

$$ \int_{t_0}^{t} (t-s)^{\alpha-1} \left[Ax(s) + \sigma(s) \frac{dw(s)}{ds} + f(s) \right] ds = t^{\alpha-1} \circ \left[Ax(t) + \sigma(t) \frac{dw(t)}{dt} + f(t) \right], $$

where $\circ$ is the convolution. The solution equation can be written as

$$ x(t) = x(t_0) + \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \circ \left[Ax(t) + \sigma(t) \frac{dw(t)}{dt} + f(t) \right]. $$

Applying the Laplace transform on both sides of above equation,

$$ X(s) = L[x(t_0); s] + \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) s^{-\alpha} \cdot AX(s) + L\left[\sigma(t) \frac{dw(t)}{dt} + f(t); s \right] $$

where $X(s)$ is the Laplace transform of $x(t)$, we have

$$ X(s) = (s^\alpha I - A)^{-1} s^\alpha L[x(t_0); s] + (s^\alpha I - A)^{-1} L\left[\sigma(t) \frac{dw(t)}{dt} + f(t); s \right] $$

Inserting the formula for laplace transform for the Mittag-Leffler function (2), we have

$$ X(s) = L[x(t_0); s] + (t^{\alpha-1} E_{\alpha,\alpha}(A(t^\alpha)) \cdot L\left[\sigma(t) \frac{dw(t)}{dt} + f(t); s \right] $$

Applying the inverse Laplace transform on both sides (see Guendouzi (2013)) and convolution, we have

$$ x(t) = x(t_0) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \times \left[Ax(t_0) + \left(\int_{t_0}^\eta \sigma(\theta) dw(\theta) \right) + f(s) \right] ds, t \in [t_0, T]. $$

Thus, the proof is completed. □

3. CONTROLLABILITY CRITERIA FOR SYSTEM

In this section, we establish the sufficient and necessary conditions of controllability criteria for impulsive system.

3.1 Linear impulsive fractional stochastic system

Consider the linear impulsive fractional stochastic differential equation is of the form as:

$$ \begin{aligned} {}^C D^\alpha x(t) = {} & Ax(t) + Gu(t) + \sigma(t) \frac{dw(t)}{dt}, \\ x(t_0) = x_0, \quad & t = [t_0, T] \setminus \{t_1, t_2, \ldots, t_\rho\}, \\ \Delta x(t_k) = {} & x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \; k = 1, 2, \ldots, \rho, \end{aligned} \tag{4} $$

where $0 < \alpha \leq 1$.

Definition 4. (Karthikeyan (2009)) The stochastic impulsive system (4) is said to be controllable on $[t_0, T]$ if, for