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Phase models and clustering in networks of oscillators with delayed, all-to-all coupling

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Abstract: We consider a general model for a network of all-to-all coupled oscillators with time delayed connections. We reduce the system of delay differential equations to a phase model where the time delay enters as a phase shift. By analyzing the phase model, we study the existence and stability of cluster solutions. These are solutions where the oscillators divide into groups; oscillators within a group are synchronized, while oscillators in different groups are phase-locked with a fixed phase difference. We show that the time delay can lead to the multistability between different cluster states. Analytical results are compared with numerical studies of the full system of delay differential equations.

Keywords: Time delay, neural network, oscillators, synchronization, stability.

1. INTRODUCTION

Many biological and physical systems can be studied using coupled oscillator models, for example neural networks (Hansel et al., 1993), laser arrays (Winful and Wang, 1988), flashing of fireflies (Mirollo and Strogatz, 1990), and movement of a slime mold (Takamatsu et al., 2000). A fundamental question about these systems is whether the elements will **phase-lock**, i.e., oscillate with some fixed phase difference, and how the physical parameters affect the answer to this question. Clustering is a type of phase locking behavior where the oscillators in a network separate into subgroups. Each subgroup consists of fully synchronized oscillators, and different subgroups oscillate with a fixed phase difference. Symmetric clustering refers to the situation when all the subgroups are the same size while non-symmetric clustering means the subgroups have different sizes.

Phase models have been used to study the behaviour of networks of coupled oscillators beginning with the work of Kuramoto (1984). While they have been used to study a variety of phenomena, especially in neural networks (Ermentrout and Kopell, 1984, 1991; Ermentrout, 1996; Galán, 2009; Hansel et al., 1993), Okuda (1993) was the first to study clustering behaviour using this tool. Considering a phase model for a network of arbitrary size with all-to-all coupling, Okuda (1993) established general criteral for the stability of all possible symmetric cluster solutions as well as some non-symmetric cluster solutions. He showed that these results gave a good prediction of stability for a variety of model networks. Recently, similar results have been obtained for networks with nearest-neighbour coupling (Miller et al., 2015). Phase model analysis has been extensively used to study phase-locking in pairs of model (Kopell and Ermentrout, 2002; Saraga et al., 2006) and experimental (Mancilla et al., 2007) neurons. More recently it has been used to study clustering in model (Kilpatrick and Ermentrout, 2011) and experimental (Galán et al., 2006) neural networks.

In many systems there are time delays in the connections between the oscillators due to the time for a signal to propagate from one element to the other. In neural networks there is a delay due to conduction of electrical activity along an axon or a dendrite and due to processing time at the synapse(Crook et al., 1997; Kopell and Ermentrout, 2002). While much work has been devoted to the study of the effect of time delays in neural networks, the majority of this work has focussed on systems where the neurons are excitatory not oscillatory, (e.g., Burić et al. (2005); Dahlem et al. (2009); Panchuk et al. (2013)), pairs of oscillators (e.g., Campbell and Kobelevskiy (2012); Kopell and Ermentrout (2002); Schuster and Wagner (1989)) or synchronization (e.g., Crook et al. (1997); Orosz (2012, 2014a)). We note that extensive work has been done on networks of Stuart-Landau oscillators with delayed coupling (e.g., Choe et al. (2010); Dahms et al. (2012)) where the model for the individual oscillators is the normal form for a Hopf bifurcation and thus the system is often amenable to direct analysis. Recent work has developed new approaches to determine the Floquet multipliers, and hence stability, of cluster solutions in delayed neural oscillator networks Orosz (2014a,b). There is also a vast literature on time delays in artificial neural networks which we do not attempt to cite here.

In this paper, we investigate the effect of time delays in the coupling on the clustering behavior of networks of all-to-all coupled identical oscillators, using the phase model approach. The advantage of this approach over Floquet analysis is that one can often draw conclusions which are independent of the particular oscillator model and the size

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of the network. The disadvantage is that phase model analysis requires weak coupling.

The plan for our article is as follows. In section 2, we review how to reduce the differential equation model for our network to a phase model. In section 3, we analyze the phase model to investigate the existence and stability of symmetric cluster states and draw some conclusions which depend only on the connectivity structure of the network. In section 4, we apply our results to a specific example: a network of all-to-all coupled Morris-Lecar oscillators with delayed synaptic coupling. A comparison of numerical results for the full model and the phase model analysis is given. In section 5, we discuss some biological implications of our results and directions for further investigation.

2. REDUCTION TO PHASE MODEL

In this section, we review how to reduce a general model for a network of all-to-all coupled oscillators with time-delayed connections to a phase model. We begin by considering the model for a single oscillator. This is a system of ordinary differential equations

$$\frac{dX}{dt} = F(X(t)), \tag{1}$$

which admits an exponentially asymptotically stable periodic orbit, denoted by $\hat{X}(t)$, with period $T = \frac{2\pi}{\Omega}$. Linearizing the model (1) about the periodic solution $\hat{X}(t)$ we obtain

$$\frac{dX}{dt} = DF(\hat{X}(t))X, \tag{2}$$

and its adjoint system

$$\frac{dZ}{dt} = -[DF(\hat{X}(t))]^T Z. \tag{3}$$

Here $DF(\hat{X}(t))$ represents the Jacobian matrix of F with respect to X, evaluated at $\hat{X}(t)$. Denote by $Z = \hat{Z}(t)$ the unique periodic solution of the adjoint system (3) satisfying the normalization condition:

$$\frac{1}{T}\int_0^T \hat{Z}(t)\cdot F(\hat{X}(t))dt = 1.$$

Now, consider the following network of identical oscillators with all-to-all, time-delayed coupling

$$\frac{dX_i}{dt} = F(X_i(t)) + \epsilon \sum_{j=1, j\neq i}^{N} G(X_i(t), X_j(t-\tau)),\ i = 1, \cdots, N. \tag{4}$$

Here G describes the coupling behavior and ϵ is referred to as the coupling strength. When ϵ is sufficiently small, we can apply the theory of weakly coupled oscillators to reduce (4) to a phase model (Ermentrout and Terman, 2010; Hoppensteadt and Izhikevich, 1997). While there are no general results on how small ϵ should be, it can been quantified for particular models. See section 4.2.

How the time delay enters into the phase model depends on the size of the delay relative to other time constants in the model. It has been shown (Ermentrout, 1994; Izhikevich, 1998; Kopell and Ermentrout, 2002) that if the delay is such that $\Omega\tau = O(1)$ with respect to the coupling strength ϵ then the appropriate model is

$$\frac{d\phi_i}{dt} = \Omega + \epsilon \sum_{j=1, j\neq i}^{N} H(\phi_j - \phi_i - \eta),\ i = 1, 2, \cdots, N, \tag{5}$$

where $\eta = \Omega\tau$. That is, the delay enters as a phase lag. The interaction function H is a 2π-periodic function which satisfies

$$H(\phi) = \frac{1}{T}\int_0^T \hat{Z}(s)G(\hat{X}(s+\phi))ds.$$

with $\hat{Z}, \hat{X}$ as defined above.

We have focussed on the case with no self-coupling, which leads to the $j \neq i$ condition in the sum above. However, the model (5) is included in the more general model

$$\frac{d\phi_i}{dt} = \tilde{\Omega} + \epsilon \sum_{j=1}^{N} H(\phi_j - \phi_i - \eta),\ i = 1, 2, \cdots, N. \tag{6}$$

For the model with no self-coupling, $\tilde{\Omega} = \Omega - H(-\eta)$, while for the model with self-coupling $\tilde{\Omega} = \Omega$. We will work with (6) in the following. Note that this model has S_N symmetry, that is, one can make any permutation of the indices of phases and the equations are left unchanged.

Finally, we note that when the delay is long enough ($\eta \sim O(1/\epsilon)$), the delay enters into the model not as phase shift, but in the argument of the oscillators, $\phi_j(t) - \phi_i(t-\tau)$ (Izhikevich, 1998). This type of model has been the subject of several studies (Kim et al., 1997; Niebur et al., 1991; Schuster and Wagner, 1989; Sethia et al., 2011; Yeung and Strogatz, 1999).

2.1 Phase difference model

Noting that the right hand side of (6) depends only on the differences between the phases, we define the variables $\theta_i = \phi_i - \phi_{i+1}$, $i = 1, 2, \ldots, N-1$. Assuming $\epsilon > 0$ and introducing the slow time $u = \epsilon t$, then gives rise to the following equations

$$\frac{d\theta_i}{du} = \sum_{j=1}^{N} \big(H(\phi_j - \phi_i - \eta) - H(\phi_j - \phi_{i+1} - \eta)\big). \tag{7}$$

Now when $j < i$, we have

$$\phi_j - \phi_i = \theta_j + \theta_{j+1} + \cdots + \theta_{i-1},$$

while, when $j > i$, we have

$$\phi_j - \phi_i = -(\theta_i + \theta_{i+1} + \cdots + \theta_{j-1}).$$

Thus, we can write (7) in the following form

$$\begin{aligned}\frac{d\theta_i}{du} &= \sum_{j=1}^{i-1} H(\sum_{k=j}^{i-1}\theta_k - \eta) + \sum_{j=i}^{N-1} H(-\sum_{k=i}^{j}\theta_k - \eta) \\ &\quad - \sum_{j=1}^{i} H(\sum_{k=j}^{i}\theta_k - \eta) - \sum_{j=i+1}^{N-1} H(-\sum_{k=i+1}^{j}\theta_k - \eta)\end{aligned} \tag{8}$$

The phase difference model (8) reduces the dimension of the model from N to $N-1$. However, it also has less symmetry than the phase model system. We will find that both models are useful in our study of cluster states.