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Auxiliary Function-based Summation Inequalities for Quadratic Functions and their Application to Discrete-time Delay Systems ⋆

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Abstract: Jensen inequality has become a powerful tool of supporting summation inequalities for quadratic functions in order to obtain stability criteria for time-delayed systems since it achieves remarkable performance with a small number of decision variables. This paper suggests a new summation inequality for quadratic functions based on an auxiliary function, which is superior to the Jensen inequality. To demonstrate the superiority of the new inequality, its application to stability analysis for discrete-time delay system is provided with a simple numerical example.

Keywords: Summation inequality, Discrete-time delay systems, Stability analysis

1. INTRODUCTION

Time delays are easily encountered in many dynamic systems, and such time delays often cause poor performance or even system instabilities Richard (2003). Thus, stability analysis of discrete-time delay systems has become an important issue and considerable research efforts have been devoted in this field.

Most recent stability analyses for discrete-time delay systems have been based on the Jensen inequality Liu and Zhang (2012); Ramakrishnan and Ray (2013); Kwon et al. (2013); Xu et al. (2013) because such approaches require fewer decision variables than approaches based on the inequality developed by Moon et al. (2001); Kim (2012) or free-weighting matrix method Gao and Chen (2007); Zhang et al. (2008); Ma et al. (2010) while achieving identical or comparable performance behavior. Recently, however, there has been an effort to analyze the conservatism of the Jensen inequality itself Briat (2011). In addition, for continuous-time systems, there have been some studies for developing an alternative inequality reducing the gap of the Jensen inequality Seuret and Gouaisbaut (2013). However, to the author's best knowledge, there are no such efforts for discrete-time systems yet.

Motivated by above observation, in this paper, we investigate to develop a new summation inequality for quadratic functions to substitute Jensen inequality in the framework of discrete-time systems. By introducing some auxiliary functions, a new summation inequality for reducing the conservatism of the Jensen inequality is developed. With an appropriate choice of the auxiliary function, the proposed summation inequality becomes easily applicable to the stability analysis for discrete-time delay systems, which yields competitive results with those of conventional approaches containing Jensen inequality.

2. PRELIMINARIES

2.1 Jensen inequality

Lemma 2.1. (Gu et al., 2003) For a positive definite matrix $R > 0$ and a vector function $\{w_i \mid i \in [a, a+n-1]\}$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n}\left(\sum_{i=a}^{a+n-1} w_i\right)^T R \left(\sum_{i=a}^{a+n-1} w_i\right). \qquad (1)$$

2.2 Lower bound lemma for reciprocal convexity

Lemma 2.2. (Park et al., 2011) Let $f_1, f_2, \ldots, f_N : \mathbf{R}^m \mapsto \mathbf{R}$ have positive values in an open subset $\mathbf{D}$ of $\mathbf{R}^m$. Then,

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the reciprocally convex combination of f_i over $\mathbf{D}$ satisfies

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i\alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j} : \mathbf{R}^m \mapsto \mathbf{R}, g_{j,i}(t) = g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}.$$

Lemma 2.1 will be referred for comparison with a new summation inequality and Lemma 2.2 will be applied in deriving a stability condition in Section 3.

3. MAIN RESULTS

In this section, we provide new summation inequalities for quadratic functions via auxiliary functions and their application to the stability analysis for discrete-time delay systems.

3.1 Auxiliary function-based summation inequalities

Theorem 1. For a positive definite matrix $R > 0$, a vector function $\{w_i \mid i \in [a, a+n-1]\}$, and an auxiliary scalar function $\{\bar{p}_i \mid i \in [a, a+n-1]\}$, the following inequality holds:

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) + \left(\sum_{i=a}^{a+n-1} \bar{p}_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right), \quad (2)$$

where

$$\sum_{i=a}^{a+n-1} \bar{p}_i = 0. \quad (3)$$

Proof. For any scalar function $\{p_i \mid i \in [a, a+n-1]\}$ and constant vector v, let us define $\{z_i \mid i \in [a, a+n-1]\}$ such as

$$z_i \triangleq w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) - p_i v,$$

which satisfies

$$\begin{aligned}
0 &\leq \sum_{i=a}^{a+n-1} z_i^T R z_i \\
&= \sum_{i=a}^{a+n-1} \left\{ w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) \right\}^T R \\
&\quad \times \left\{ w_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} w_j \right) \right\} + \left(\sum_{i=a}^{a+n-1} p_i^2 \right) v^T R v \\
&\quad -2v^T R \left\{ \sum_{i=a}^{a+n-1} \left(p_i - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} p_j \right) \right) w_i \right\} \\
&= \sum_{i=a}^{a+n-1} w_i^T R w_i - \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) \\
&\quad + \left(\sum_{i=a}^{a+n-1} p_i^2 \right) [v - \bar{v}]^T R [v - \bar{v}] \\
&\quad - \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right),
\end{aligned}$$

where

$$\bar{p}_i = p_i - \frac{1}{n} \sum_{j=a}^{a+n-1} p_j,$$

$$\bar{v} = \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right).$$

Let us choose $v = \bar{v}$, which yields

$$\sum_{i=a}^{a+n-1} w_i^T R w_i \geq \frac{1}{n} \left(\sum_{i=a}^{a+n-1} w_i \right)^T R \left(\sum_{i=a}^{a+n-1} w_i \right) + \left(\sum_{i=a}^{a+n-1} p_i^2 \right)^{-1} \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right)^T R \left(\sum_{i=a}^{a+n-1} \bar{p}_i w_i \right). \quad (4)$$

Since it holds that

$$\sum_{i=a}^{a+n-1} \bar{p}_i^2 = \sum_{i=a}^{a+n-1} p_i^2 - \frac{1}{n} \left(\sum_{j=a}^{a+n-1} p_j \right)^2 \leq \sum_{i=a}^{a+n-1} p_i^2, \quad (5)$$

the choice of p_i as $\bar{p}_i$ will produce the tightest lower bound of (4) as the form of (2). Then, the resulting $\bar{p}_i$ should yield the kind of zero mean condition (3), which completes the proof. ∎

Remark 1. Since the last term in (2) is a positive quantity from the condition $R > 0$, the proposed inequality (2) is much tighter than Jensen inequality (1). Therefore, Theorem 1 can be applied to the stability analysis of discrete-time delay systems for reducing the conservatism of Jensen inequality.

Remark 2. For the application of Theorem 1, the design of auxiliary function $\bar{p}_i$ is essential. Let us determine α in the form of an auxiliary function $\bar{p}_i = (i-a+1) - \alpha$. From the condition (3),

$$\sum_{i=a}^{a+n-1} \{(i-a+1) - \alpha\} = \frac{n(n+1)}{2} - n\alpha = 0,$$

which implies that $\alpha = (n+1)/2$. Then we have that

$$\begin{aligned}
\sum_{i=a}^{a+n-1} \bar{p}_i^2 &= \sum_{i=a}^{a+n-1} \left\{ (i-a+1) - \frac{n+1}{2} \right\}^2 \\
&= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{2} + \frac{n(n+1)^2}{4}
\end{aligned}$$