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A Novel Approach to Robust Stability Analysis of Linear Time-Delay Systems ⋆

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Abstract: In this paper, we demonstrate that the Lyapunov–Krasovskii functional, which does not admit a quadratic lower bound, can be applied for the robust exponential stability analysis as well as for obtaining the exponential estimates for the solutions of linear time-invariant differential-difference systems with multiple delays.

Keywords: time-delay systems, Lyapunov–Krasovskii functionals, exponential stability, robust stability, exponential estimates, decay rate

1. INTRODUCTION

The Lyapunov–Krasovskii approach to stability analysis of time-delay systems is based on the well-known Krasovskii theorem (Krasovskii, 1956). Roughly speaking, it states that if there is a positive definite functional, whose time-derivative along the solutions of the system is negative definite, then the system is asymptotically stable. One of the possible ways to use this theorem is to prescribe the (negative definite) time-derivative and then to construct the functional that has the same derivative along the solutions. For linear time-invariant delay systems, the functionals with a prescribed derivative have been developed in the works of Repin (1965), Infante & Castelan (1978), Huang (1989) and Kharitonov & Zhabko (2003). In these papers, the structure, the existence issue, and further the explicit form of the functionals and their positive definiteness have been studied. As a result, two functionals satisfying the Krasovskii theorem were constructed. The first one was proposed in Huang (1989), let us denote it by v_0; its time-derivative along the solutions of the system is the quadratic form of the current state $x(t)$. In its turn, the second one's derivative is the functional depending on the whole delay system's state x_t. The second functional was introduced in Kharitonov & Zhabko (2003) and was called the functional of the complete type. The important point is that the functional v_0 does not admit a quadratic lower bound and admits only the local cubic one if the system is exponentially stable (see Huang, 1989), whereas the complete-type functional admits the quadratic bound and, therefore, is effective in applications. There are many contributions addressing the applications of the complete-type functionals, see, for instance, Egorov & Mondiè (2014), Jarlebring et al. (2011), Ochoa et. al (2013), and Kharitonov (2013). The applications important for us in this paper are the robustness analysis (Kharitonov & Zhabko, 2003) and the construction of the exponential estimates for the solutions (Kharitonov & Hinrichsen, 2004), as for linear time-invariant delay systems the asymptotic stability is equivalent to the exponential one. On the contrary, the functional v_0 is considered to be not suitable for solving the problems of this kind.

However, in the works Zhabko & Medvedeva (2011), Medvedeva & Zhabko (2013; 2015) the following has been shown. In spite of the fact that the functional v_0 does not admit a quadratic lower bound on the set of arbitrary continuous functions, it admits such a bound on the set of functions satisfying the condition $\|\varphi(\theta)\| \leqslant \|\varphi(0)\|$, $\theta \in [-h, 0]$, where h is the maximal delay, if the system is exponentially stable. In terms of such bound, the exponential stability criterion was established, and the constructive approach for the stability analysis was developed.

The aim of the present paper is to demonstrate that the functional v_0 can be effective not only in the stability but also in the robust stability analysis as well as in the construction of the exponential estimates for the solutions of the exponentially stable systems. In other words, we are going to show the possibility to analyze the robustness and to estimate the decay rate and the γ-factor (see Definition 1) without making use of the complete-type functionals. Our approach is based on the above-mentioned exponential stability criterion. The special integral estimate for the derivative of the functional plays a key role as well.

It is worth pointing out that there is a great variety of works where the problems we address are treated on the basis of the LMI approach, see, for instance, Mondiè & Kharitonov (2005) or the survey papers Kharitonov (1999) and Niculescu et al. (1997). In Bellman & Cooke (1963) the exponential estimates for the solutions are constructed directly in terms of the Laplace transform.

The paper is organized as follows. Section 2 contains the preliminaries. Then, Section 3 is devoted to the robustness analysis whilst the exponential estimates for the solutions are provided in Section 4. In Section 5, we illustrate the work with examples comparing our results with those obtained in Kharitonov & Zhabko (2003) and Kharitonov & Hinrichsen (2004) by use of the complete-type functionals.

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2. PRELIMINARY RESULTS

In this paper, we consider a time-delay system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j), \tag{1}$$

where $A_j \in \mathbb{R}^{n\times n}$, $j = \overline{0, m}$, are the constant matrices, and $0 = h_0 < h_1 < \ldots < h_m = h$ are the constant delays. We use the standard notation: $x(t, \varphi)$, or briefly $x(t)$, denotes the solution of system (1) with the piecewise continuous initial function φ, i.e. $\varphi \in PC\big([-h, 0], \mathbb{R}^n\big)$; then, $x_t(\varphi)$, or briefly x_t, stands for the segment of the solution

$$x_t(\varphi): \ \theta \to x(t + \theta, \varphi), \ \theta \in [-h, 0].$$

On the space of the piecewise continuous functions the uniform norm

$$\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$$

is defined, where $\|\cdot\|$ is the euclidian norm.

Definition 1. (Bellman & Cooke, 1963) System (1) is called *exponentially stable*, if there exist $\gamma \geqslant 1$ and $\sigma > 0$ such that

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t} \|\varphi\|_h, \quad t \geqslant 0,$$

for every solution of system (1).

Given a positive definite matrix W, the functional satisfying the condition

$$\frac{dv_0(x_t)}{dt} = -x^T(t)Wx(t), \quad t \geqslant 0,$$

along the solutions of system (1) is of the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\ &+ 2\varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U(-\theta - h_j)A_j\varphi(\theta)d\theta \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0} \varphi^T(\theta_1)A_k^T \\ &\times\Big(\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j\varphi(\theta_2)d\theta_2\Big)d\theta_1, \end{aligned} \tag{2}$$

see Huang (1989); Kharitonov & Zhabko (2003). Here $U(\tau)$ is the Lyapunov matrix, associated with W, i.e. the solution of the following set of the matrix equations

$$U'(\tau) = \sum_{j=0}^{m} U(\tau - h_j)A_j, \ \tau \geqslant 0;$$

$$U(-\tau) = U^T(\tau), \ \tau \geqslant 0;$$

$$\sum_{j=0}^{m}\big[U(-h_j)A_j + A_j^T U(h_j)\big] = -W.$$

The Lyapunov matrix and, therefore, functional (2) exists for any symmetric matrix W, if and only if the so-called Lyapunov condition holds: the system does not have an eigenvalue s such that $-s$ is also an eigenvalue, see Kharitonov (2013). The Lyapunov matrix is continuous.

Functional (2) admits the following upper bound:

Lemma 2. (Kharitonov, 2013) If the Lyapunov condition holds, then

$$\big|v_0(\varphi)\big| \leqslant \eta\|\varphi\|_h^2, \ \varphi \in PC\big([-h, 0], \mathbb{R}^n\big),$$

where

$$\eta = M\alpha^2, \ M = \max_{\tau\in[0,h]} \|U(\tau)\|, \ \alpha = 1 + \sum_{j=1}^{m} \|A_j\|h_j.$$

As for a lower bound, there is only the local cubic one: If system (1) is exponentially stable, then for every H there exists $\kappa > 0$ such that

$$v_0(\varphi) \geqslant \kappa\|\varphi(0)\|^3, \quad \|\varphi\|_h \leqslant H,$$

here φ is a continuous function, see Huang (1989). Nevertheless, on the special set of functions

$$S = \big\{\varphi \in PC\big([-h, 0], \mathbb{R}^n\big) \,\big|\, \|\varphi(\theta)\| \leqslant \|\varphi(0)\|, \, \theta \in [-h, 0]\big\}$$

functional (2) admits a quadratic lower bound, as the following criterion states.

Theorem 3. (Zhabko & Medvedeva, 2011; 2015) Given a positive definite matrix W, system (1) is exponentially stable, if and only if there exists a functional $v_0(\varphi)$ such that the following conditions hold:

1. $\dfrac{dv_0(x_t)}{dt} = -x^T(t)Wx(t)$;
2. there exists $\mu > 0$ such that
$$v_0(\varphi) \geqslant \mu\|\varphi(0)\|^2, \quad \varphi \in S.$$

Note that in the necessity part of Theorem 3 the constant μ is obtained constructively:

$$\mu = \frac{\lambda_{\min}(W)\delta}{4},$$

where $\lambda_{\min}(W)$ is the minimal eigenvalue of W, and $\delta > 0$ is the solution of the equation

$$\alpha K e^{K\delta} = \frac{1}{2\delta},$$

here $K = \sum_{j=0}^{m} \|A_j\|$, and α is defined in Lemma 2.

3. ROBUST STABILITY ANALYSIS

In this section, we consider the same problem statement as in Kharitonov & Zhabko (2003), see also Kharitonov (2013). Assume that system (1) is exponentially stable and define the following perturbed system

$$\dot{y}(t) = \sum_{j=0}^{m}(A_j + \Delta_j)y(t - h_j). \tag{3}$$

Here the constant matrices Δ_j are such that

$$\|\Delta_j\| \leqslant \rho_j, \quad j = \overline{0, m}, \tag{4}$$

where ρ_j are the constant values. Our aim is to find the conditions on these values under which system (3) remains exponentially stable.

Following Kharitonov & Zhabko (2003), for the stability analysis of system (3) we will use functional (2), corresponding to system (1). The time-derivative of this functional along the solutions of system (3) is of the form

$$\frac{dv_0(y_t)}{dt} = -y^T(t)Wy(t) + l(y_t), \tag{5}$$

where

$$l(y_t) = 2\Big[\sum_{j=0}^{m}\Delta_j y(t - h_j)\Big]^T \times$$

$$\times\Big[U(0)y(t) + \sum_{k=1}^{m}\int_{-h_k}^{0} U(-\theta - h_k)A_k y(t + \theta)d\theta\Big],$$