Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-12 (2015) 245–250

The delay Lyapunov matrix in robust stability analysis of time-delay systems

Alexey V. Egorov * **Sabine Mondié** **

* *Saint-Petersburg State University, Saint-Petersburg, Russia (e-mail: alexey3.1416@gmail.com)*
** *Departamento de Control Automático, Cinvestav, IPN, México D.F. (e-mail: smondie@ctrl.cinvestav.mx)*

Abstract: The maximum of the norm of the delay Lyapunov matrix function of exponentially stable linear time-delay systems is proven to be achieved at zero. We apply this result to the robust stability analysis of systems with multiple delays. We consider the cases of uncertain matrices of the system and uncertain delays.

Keywords: time-delay system; robust stability; Lyapunov-Krasovskii functional; Lyapunov matrix; uncertain delay.

1. INTRODUCTION

The method of Lyapunov functions is widely used in the theory of delay free differential systems. The method allows to construct exponential estimates for the solutions, to analyze the stability and robust stability of systems. To obtain similar results for linear time-delay systems, we need to find a universal Lyapunov-Krasovskii functional, satisfying Krasovskii theorem whenever the corresponding system is exponentially stable. Based on the results obtained in Repin (1965), Infante and Castelan (1978), Huang (1989), such functionals, called complete type functionals, were introduced in Kharitonov and Zhabko (2003). They are based on the special delay Lyapunov matrix, which is a generalization of the Lyapunov matrix for delay free systems.

In Zhabko and Medvedeva (2011), Mondié (2012), Egorov (2014), Medvedeva and Zhabko (2015), Egorov and Mondié (2014, Automatica) it was shown how to use the functionals to check the stability or instability of systems. The book of Kharitonov (2013) considers the construction of exponential estimates for the solutions, and the problem of robust stability of perturbed systems with uncertain matrices. Applying these methods to systems of large dimension or to systems with several delays is associated with a certain computational complexity. In this paper, we derive a new property of the delay Lyapunov matrix that simplifies significantly the computations.

In the paper we also deal with the case of uncertain delays. A number of papers are devoted to this topic, see, for example, Park (1999), Kolmanovskii and Richard (1999). The functionals with prescribed derivatives have been successfully applied to this problem in Kharitonov and Niculescu (2003), Fridman (2006). In the present contribution we consider the general system with multiple delays, and show a way how to reduce the computational complexity of the results.

The contribution is organized as follows: the main definitions are reminded in section 2, the basic results on the delay Lyapunov matrix and its applications are listed in section 3. New properties of the delay Lyapunov matrix are proven in section 4, section 5 is devoted to the stability of systems with uncertain delays. Two illustrative examples and concluding remarks end the note.

2. PRELIMINARIES

Consider the linear system with multiple constant delays:

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-h_j), \quad t > 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, A_j are constant matrices, and $0 = h_0 \leqslant h_1 \leqslant \ldots \leqslant h_m = H$ are ordered delays.

The initial condition is

$$x(\theta) = \varphi(\theta), \quad \theta \in [-H, 0], \ \varphi \in \mathcal{D} = PC([-H,0], \mathbb{R}^n).$$

For a given initial function φ:

$$x_t(\varphi) : \theta \to x(t+\theta, \varphi), \quad \theta \in [-H, 0],$$

denotes the restriction of the solution $x(t,\varphi)$ of system (1) to the interval $[t-H, t]$. When the initial condition is not crucial, the argument φ is omitted.

Notation $Q > 0$ ($Q \geqslant 0$) means that the symmetric matrix Q is positive definite (semidefinite). We write $Q > S$ ($Q \geqslant S$), if the matrix $Q - S$ is positive definite (semidefinite). In the paper the Euclidian norm for vectors and matrices is denoted by $\|\cdot\|$:

$$\|b\| = \sqrt{b_1^2 + \ldots + b_n^2},$$

$$\|A\| = \sup_{b \neq 0} \frac{\|Ab\|}{\|b\|} = \sqrt{\lambda_{\max}(A^T A)},$$

where $b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $\lambda_{\max}(A^T A)$ is the maximal eigenvalue of $A^T A$. Note that

$$\|S\| = \lambda_{\max}(S),$$

if S is positive definite. For piecewise continuous functions the supremum norm $\|\varphi\|_\infty = \sup_{\theta \in [-H,0]} \|\varphi(\theta)\|$ is used.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.385

System (1) is said to be *exponentially stable* if there exist constants $\gamma \geqslant 0$ and $\sigma > 0$, such that

$$\|x(t,\varphi)\| \leqslant \gamma e^{-\sigma t}\|\varphi\|_{\mathcal{H}}, \quad t \geqslant 0,\ \varphi \in \mathcal{D}.$$

In this paper we will use the sign function $\operatorname{sign}(\theta)$ which is equal to $\theta/|\theta|$, when $\theta \neq 0$, and is 0 at zero. We also use the Heaviside step function χ which is equal to 1 for every nonnegative argument.

3. THE DELAY LYAPUNOV MATRIX IN ROBUST STABILITY

The continuous function $U(\tau) \in \mathbb{R}^{n\times n}$, $\tau \in [-H, H]$ is called *the delay Lyapunov matrix*, associated with the positive definite matrix W, if it satisfies the equations

$$U'(\tau) = \sum_{j=0}^{m} U(\tau - h_j)A_j, \quad \tau > 0, \tag{2}$$

$$U(\tau) = U^T(-\tau), \tag{3}$$

$$\sum_{j=0}^{m}\left[U(-h_j)A_j + A_j^T U(h_j)\right] = -W. \tag{4}$$

The construction of the delay Lyapunov matrix reduces to the calculation of a matrix exponential in case of commensurate delays (see, Kharitonov (2013)). Otherwise, the exists a number of numerical schemes (see, Kharitonov (2013), Huesca et al. (2009), Jarlebring et al. (2011)).

The delay Lyapunov matrix allows to construct the *complete type functional*:

$$v(\varphi) = v_0(\varphi) + \sum_{j=1}^{m}\int_{-h_j}^{0} \varphi^T(\theta)\left(W_j + (\theta + h_j)W_{m+j}\right)\varphi(\theta)\,d\theta, \tag{5}$$

where

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\ &+ 2\varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U^T(\theta + h_j)A_j\varphi(\theta)\,d\theta + \sum_{k=1}^{m}\int_{-h_k}^{0}\varphi^T(\theta_1) \\ &\cdot A_k^T\left(\sum_{j=1}^{m}\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j\varphi(\theta_2)\,d\theta_2\right)d\theta_1, \end{aligned}$$

and W_j, $j = \overline{0, 2m}$, are positive definite matrices, such that $W_0 + \sum_{j=1}^{m}(W_j + h_jW_{m+j}) = W$. It has been shown in Kharitonov and Zhabko (2003) that the functional admits a positive lower bound when the corresponding system is exponentially stable, i.e., there exists a number $\alpha > 0$, such that $v(\varphi) \geqslant \alpha\|\varphi(0)\|^2$, $\varphi \in \mathcal{D}$.

Among the various applications of complete type functionals, we consider two, concerning robust stability.

Problem 1. In the third chapter of Kharitonov (2013) the author considers perturbed system of the form

$$\dot{y}(t) = \sum_{j=0}^{m}\Big(A_j + \Delta_j(t)\Big)\,y(t - h_j), \tag{6}$$

where $\Delta_j(t)$, $j = \overline{0, m}$, are continuous bounded matrix valued functions on $[0, \infty)$. Introduce the following notation:

$$\nu = \max_{\tau\in[-H,\,H]}\|U(\tau)\|, \tag{7}$$

$$a_j = \|A_j\|,\ j = \overline{0, m}, \quad \lambda_{\min} = \min_{j=\overline{0,2m}}\{\lambda_{\min}(W_j)\}.$$

Theorem 1. (Kharitonov (2013)). If system (1) is exponentially stable, and for any $t \geqslant 0$

$$\|\Delta(t)\| = \sqrt{\sum_{j=0}^{m}\|\Delta_j(t)\|^2} < \frac{\lambda_{\min}}{2\nu}\left(1 + \sum_{j=1}^{m} h_j a_j^2\right)^{-\frac{1}{2}}, \tag{8}$$

then system (6) is also exponentially stable.

Note that it is computationally difficult to find the maximal value of $\|U(\tau)\|$ on $[-H, H]$, especially if $n > 1$. In the next section we will show that the task can be reduced to the computation of $\|U(0)\|$.

Problem 2. The paper by Kharitonov and Niculescu (2003) was devoted, in particular, to the stability analysis of system

$$\dot{y}(t) = A_0y(t) + A_1y(t - h - \eta). \tag{9}$$

Theorem 2. (Kharitonov and Niculescu (2003)). System (9) is exponentially stable, if system

$$\dot{x}(t) = A_0x(t) + A_1x(t - h)$$

is exponentially stable, and the number η satisfies the following inequalities:

$$\begin{cases} |\eta| < \dfrac{h}{3}, \\ W_0 > |\eta|U(0)[M_1^{-1} + M_2^{-1}]U(0), \\ W_2 > 2|\eta|A_1^TU(\tau)[M_3^{-1} + M_4^{-1}]U^T(\tau)A_1, \end{cases} \tag{10}$$

for every $\tau \in [0, H]$. Here W_0, W_2, M_i, $i = \overline{1, 4}$, are some positive definite matrices, such that

$$W > W_0 + 3hW_2,$$
$$W_2 > 2(A_1^2)^T[M_2 + hM_4]A_1^2,$$
$$W_2 > 2A_0^TA_1^T[M_1 + hM_3]A_1A_0.$$

Unfortunately, inequalities (10) are quite difficult to verify, because they contain a continuous parameter τ. In Section 5, where we consider the multiple delay case, this problem is overcome.

4. NORM OF THE DELAY LYAPUNOV MATRIX

We start with the following necessary stability condition.

Theorem 3. (Egorov and Mondié (2014, Automatica)). If system (1) is exponentially stable, then

$$\begin{pmatrix} U(0) & U(\tau) \\ U^T(\tau) & U(0) \end{pmatrix} > 0, \quad \tau \in (0, H]. \tag{11}$$

Using this result, we are able to prove that the maximal value of the norm of the delay Lyapunov matrix is achieved at zero.

Lemma 4. If system (1) is exponentially stable, then

$$\|U(\tau)\| < \|U(0)\|, \quad \tau \in (0, H].$$

Proof. Using Schur complement one can show that inequality (11) is equivalent to

$$U(0) > 0,$$