A least-square spectral element method for stability analysis of time delay systems [★]

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Abstract: This paper presents a numerical method for the stability analysis of retarded functional differential equations with time-periodic coefficients. The method approximates the solution segments, corresponding to the end points of the principal period, by their piece-wise Lagrange interpolants. Then a mapping between these solution segments is obtained by the minimization of the least-square integral of the residual error. Finally, stability properties are determined using the matrix approximation of the monodromy operator, obtained by this mapping, according to the Floquet theorem. The formulation of the method is presented for an equation of general type while results are shown for the delayed oscillator.

Keywords: time delay, time-varying system, stability analysis, stability domains, least-squares method, numerical method, spectral element

1. INTRODUCTION

In the recent decades the significance of time-delay has been discovered in an increasing number of applications in engineering and biosciences. Machine tool vibrations (**??**), wheel dynamics of vehicles (**?**), traffic dynamics (**?**), control (**?**), human balancing (**?**), population dynamics (**?**) and epidemiology (**?**) can be mentioned as examples. In the majority of the above applications it is desired to keep the system in the proximity of a particular state or motion by a proper tuning of system parameters. Therefore, stability properties of time-delay systems has been given an increasing attention and, as a result, many analytical and numerical methods can be found on their linear stability analysis. A commonly used analytical method is the D-subdivision method (**?**), while examples for numerical methods are the semi-discretization method (**?**), spectral element method (**?**), pseudospectral collocation method (**?**) or methods based on the truncation of Hill's infinite matrix (e.g. **??**).

Although time dependency of the system parameters are often neglected, therefore the particular phenomenon is described by time-invariant (autonomous) differential equations, this simplification cannot be justified in all cases. For instance, milling operations, turning operations with spindle speed variation or digitally controlled machines are inevitably time-varying systems. The governing equations for these systems are usually modelled as differential equations with time-periodic parameters, and their stability can be determined using the Floquet theory.

This paper deals with the formulation of a method for the numerical stability analysis of linear retarded functional differential equations (RFDE) with time-periodic coefficients. First, the main steps of the method are presented

for a RFDE of general type. Then it is applied to a particular example: the delayed oscillator. For this example stability charts are constructed and the convergence of stability boundaries are demonstrated. Finally, based on these results, some conclusions are drawn.

2. METHOD

In this section the formulation of the method is presented for a RFDE involving point delays and distributed delay with periodic coefficients of principal period T, having the form

$$\dot{\mathbf{y}}(t)=\mathbf{A}(t)\mathbf{y}(t)+\sum_{l=1}^{\sigma}\mathbf{B}_l(t)\mathbf{y}(t-\tau_l)+\int_{-a}^{-b}\boldsymbol{\gamma}(t,\theta)\mathbf{y}(t+\theta)\mathrm{d}\theta, \quad (1)$$

where $\mathbf{y}:\mathbb{R}\to\mathbb{R}^n$; $\mathbf{A},\mathbf{B}_l:\mathbb{R}\to\mathbb{R}^{n\times n}$; $0<\tau_l$, $l=1,2,\ldots,\sigma$; $0\le b<a$; $\boldsymbol{\gamma}:\mathbb{R}^2\to\mathbb{R}^{n\times n}$; and $\mathbf{A}(t)=\mathbf{A}(t+T)$; $\mathbf{B}_l(t)=\mathbf{B}_l(t+T)$, $l=1,2,\ldots,\sigma$; $\boldsymbol{\gamma}(t,\theta)=\boldsymbol{\gamma}(t+T,\theta)$, $\forall\theta$ for all t. The mapping between a solution segment $\mathbf{y}^{-\tau+t_1,t_1}$ and an initial function $\mathbf{y}^{-\tau,0}$ is given by the solution operator as $\mathbf{y}^{-\tau+t_1,t_1}=\boldsymbol{\mathcal{U}}(t_1)\mathbf{y}^{-\tau,0}$, where $\tau>\max\{a,\tau_1,\tau_2,\ldots\tau_\sigma\}$. Here and in the following, notation $\mathbf{y}^{a,b}$ refers to $\{\mathbf{y}(t):[a,b]\}$. According to the Floquet theory the stability of (1) can be determined from the eigenvalues of the monodromy operator $\boldsymbol{\mathcal{U}}(T)$. Namely, (1) is stable if and only if all the eigenvalues of $\boldsymbol{\mathcal{U}}(T)$ have modulus less than one (see Chapter 8 in **?** for details). Although there exist some counterexamples (see e.g. (**?**)), in general, the eigenvalues of the monodromy operator cannot be determined in closed form, therefore an approximation of $\boldsymbol{\mathcal{U}}(T)$ is necessary. In this section a method is presented for the calculation of the matrix approximation $\tilde{\mathbf{U}}$ of $\boldsymbol{\mathcal{U}}(T)$, for which the above stability condition can easily be checked.

After performing numerical integration using Gaussian quadrature, (1) can be approximated by the RFDE

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) + \sum_{p=1}^{m} \mathbf{B}_p(t)\mathbf{y}(t - \tau_p) \tag{2}$$

which contains point delays only. Here

$$\mathbf{B}_p(t) = \tfrac{a-b}{2}\boldsymbol{\gamma}(t, \theta_{p-\sigma})w_{p-\sigma}, \tag{3}$$

$\tau_p = -\theta_{p-\sigma}$ for $p = \sigma+1, \sigma+2, \ldots, m$ and $\theta_q = \frac{a-b}{2}\eta_q - \frac{a+b}{2} \in [-a, -b]$, where $\{\eta_q\}_{q=1}^{\rho} \in [-1, 1]$ are the set of quadrature nodes with $m = \sigma + \rho$. For convenience $\tau_1 \le \tau_2 \le \ldots \le \tau_m$ is assumed. In the following the approximate system (2) is studied.

Consider the solution segment $\mathbf{y}^{-\tau,T}$ where $\tau = KT$ and

$$K = \begin{cases} [\tau_m/T] & \text{if } \tau_m \bmod T = 0 \\ [\tau_m/T] + 1 & \text{otherwise} \end{cases} \tag{4}$$

with $[\cdot]$ denoting the integer part. Define the linear operator $\mathcal{A}$ as

$$\mathcal{A}\mathbf{y} = \left\{ \dot{\mathbf{y}}(t) - \mathbf{A}(t)\mathbf{y}(t) - \sum_{p=1}^{m} \mathbf{B}_p(t)\mathbf{y}(t - \tau_p) : t \in (0, T] \right\} \tag{5}$$

which maps the solution segment $\mathbf{y}^{-\tau,T}$ to zero. Split the solution segment $\mathbf{y}^{-\tau,T}$ onto $(K+1)E$ number of equidistant subsegments (referred to as elements in the sequel) as

$$\mathbf{y}_k = \mathbf{y}^{(k-1)h,kh}, \tag{6}$$

$k = -EK+1, -EK+2, \ldots, E$, where $h = T/E$ is the length of elements. The connection between these elements are defined by conditions

$$\mathbf{y}_k(kh) = \mathbf{y}_{k+1}(kh) \tag{7}$$

$k = -EK+1, -EK+1, \ldots, E-1$. The splitting of solution segment $\mathbf{y}^{-\tau,T}$ transforms (2) into the system of differential equations

$$\dot{\mathbf{y}}_k(t) - \mathbf{A}(t)\mathbf{y}_k(t) - \sum_{p=1}^{m} \mathbf{B}_p(t)\mathbf{y}_{k,p}^*(t), \tag{8}$$

where $t \in ((k-1)h, kh]$ and $k = 1, 2, \ldots, E$, with boundary conditions (7) and

$$\mathbf{y}_{k,p}^*(t) = \begin{cases} \mathbf{y}_{k-r_p-1}(t) & \text{if } (t-\tau_p) \in ((k-r_p-2)h, (k-r_p-1)h] \\ \mathbf{y}_{k-r_p}(t) & \text{if } (t-\tau_p) \in ((k-r_p-1)h, (k-r_p)h] \end{cases} \tag{9}$$

where $r_p = [\tau_p/h]$. System (8) is equivalent to the system of operator equations

$$\mathcal{S}^k\mathbf{y}_k - \sum_{p=1}^{m} \mathcal{Q}^{k,p}\mathbf{y}_{k-r_p-1} - \sum_{p=1}^{m} \mathcal{R}^{k,p}\mathbf{y}_{k-r_p} = \mathbf{0}, \tag{10}$$

$k = 1, 2, \ldots, E$, where the operators are defined as

$$\mathcal{S}^k\mathbf{y}_k = \begin{cases} \dot{\mathbf{y}}_k(t) - \mathbf{A}(t)\mathbf{y}_k(t) & t \in ((k-1)h, kh] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{11}$$

$$\mathcal{Q}^{k+r_p+1,p}\mathbf{y}_k = \begin{cases} \mathbf{B}_p(t+\tau_p)\mathbf{y}_k(t-\tau_p) & (t-\tau_p) \in (kh - \alpha_p, kh] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{12}$$

$$\mathcal{R}^{k+r_p,p}\mathbf{y}_k = \begin{cases} \mathbf{B}_p(t+\tau_p)\mathbf{y}_k(t-\tau_p) & (t-\tau_p) \in ((k-1)h, kh - \alpha_p] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{13}$$

with $\alpha_p = \tau_p \bmod h$. By the introduction of new element-wise coordinate $\zeta = 2(t-(k-1)h)/h-1$ the above operators have the form

$$\mathcal{S}^k\mathbf{y}_k =$$
$$\begin{cases} \frac{2}{h}\mathbf{y}_k'(\zeta) - \mathbf{A}\left(\frac{h(\zeta+1)}{2}+(k-1)h\right)\mathbf{y}_k(\zeta) & \zeta \in (-1, 1] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{14}$$

$$\mathcal{Q}^{k+r_p+1,p}\mathbf{y}_k =$$
$$\begin{cases} \mathbf{B}_p\left(\frac{h(\zeta+1)}{2}+(k+r_p)h\right)\mathbf{y}_k(\zeta+2-\beta_p) & \zeta \in (-1, -1+\beta_p] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{15}$$

$$\mathcal{R}^{k+r_p,p}\mathbf{y}_k =$$
$$\begin{cases} \mathbf{B}_p\left(\frac{h(\zeta+1)}{2}+(k+r_p-1)h\right)\mathbf{y}_k(\zeta-\beta_p) & \zeta \in (-1+\beta_p, 1] \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{16}$$

where $\beta_p = 2\alpha_p/h$. Due to this coordinate transformation boundary conditions (7) are given by

$$\mathbf{y}_k(1) = \mathbf{y}_{k+1}(-1) \tag{17}$$

$k = -EK+1, -EK+2, \ldots, E-1$. Note that $\mathbf{y}^{-\tau,T}$ satisfies the system of operator equations (10) with boundary conditions (17) and satisfies

$$\mathcal{A}\mathbf{y} = \mathbf{0}, \tag{18}$$

also. Therefore, the uniqueness of the solutions of both (10) and (18) would result the equivalence of these two problems.

For a given initial function $\mathbf{y}^{-\tau,0}$ operator equation (18) can be reformulated as the variational problem

$$\delta\mathcal{I}(\mathbf{y}^{0,T}) = \mathbf{0}, \tag{19}$$

where the minimum of functional

$$\mathcal{I}(\mathbf{y}^{0,T}) = \|\mathcal{A}\mathbf{y}\|_{L^2} \tag{20}$$

is sought with $\|\cdot\|_{L^2}$ being the L^2 norm over domain $t \in [0, T]$, defined by the usual scalar product

$$\langle f, g \rangle = \int_0^T f(t)g(t)\mathrm{d}t. \tag{21}$$

The solution of (18) gives the global minimum of (20), therefore the uniqueness of the solutions of (18) and (19) would result the equivalence of these two problems. Assuming equivalence between (10) and (18), (19) can be given as

$$\delta\left(\sum_{k=1}^{E} \left\| \mathcal{S}^k\mathbf{y}_k - \sum_{p=1}^{m} \mathcal{Q}^{k,p}\mathbf{y}_{k-r_p-1} - \sum_{p=1}^{m} \mathcal{R}^{k,p}\mathbf{y}_{k-r_p} \right\|_{L^2} \right) = \mathbf{0}, \tag{22}$$

where the scalar product is now defined on $\zeta \in (-1, 1]$.

This numerical method, discretizes the monodromy operator via the approximation of the solution segment $\mathbf{y}^{-\tau,T}$ by its piece-wise (element-wise) Lagrange interpolant. Therefore, the approximation of the elements read as

$$\tilde{\mathbf{y}}_k(\zeta) = \sum_{j=1}^{N} \phi_j(\zeta)\mathbf{y}_{k,j}, \tag{23}$$

$k = -EK+1, -EK+2, \ldots, E$, where $\mathbf{y}_{k,j} = \mathbf{y}_k(\zeta_j)$ and $\phi_j(\zeta)$ are the Lagrange base polynomials. The node