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Viscosity Solution of Bellman-Isaacs Equation Arising in Non-linear Uncertain Object Control

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Abstract: The problem of optimal control for a class of non-linear objects with uncontrolled bounded disturbances is formulated in the sense of a differential game. In case of problems with quadratic quality functional, the problem of optimal control search is reduced to finding of solution of Hamilton-Jacobi-Isaacs equation. Solutions of this equation at the rate of functioning of the object are searched by means of special algorithmic procedures obtained with the use of viscosity solution theory. The obtained results may be used for solving of theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, engineering, control and navigation. *This work (research grant №14-01-0112) was supported by The National Research University Higher School of Economics' Acad. Fund Program.*

Keywords: Non-linear systems, differential games, Hamilton-Jacobi-Isaacs equation, viscosity solution

1. INTRODUCTION

Successful implementation of obtained theoretical results in a number of problems is connected with solving of partial first-order derivative equations. Such partial derivative equations appear under solving of a great number of theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, engineering, control, etc. Such equations are Hamilton-Jacobi equation in theoretical mechanics (Arnold, 1977), Bellman equation in theory of optimal control (Bellman, 1957), Isaacs equation (Isaacs, 1965), eikonal equation in geometrical optics (Courant, 1961), Burgers and Hopf limit equations in gas dynamics and hydrodynamics (Bardi and Capuzzo-Dolcetta, 1997, Crandal, 1992), etc.
The method of characteristics proposed in the first half of the 19^{th} century by O. Cauchy for solving boundary problems for such equations reduces integrating of partial first-order derivative equations to integrating of a system of ordinary differential equations. This method is based on the fact that invariance of graph of the classical solution for a boundary problem is relative to the characteristics. However, in case of partial derivative nonlinear equation, smooth solution exists only locally.
In 1950-1970s a lot of mathematicians paid much attention to generalized solutions of Hamilton-Jacobi and other types of equations (Evans, 1998; Bardi and Capuzzo-Dolcetta, 1997). Developed methods mainly based on integral methods and integral properties of generalized solutions.
In early 1980s a concept of viscosity solution was introduced the existence of which was proved by method of disappearing viscosity (Crandall et al., 1992). The method is also being developed at present time. The researches pay attention to analytical, constructive and numerical methods of construction of viscosity solutions (Cacace et al., 2011) and application of theoretical results to solving of various applied problems. Another well-known concept of the generalized solution based on idempotent analysis was proposed in works by V.P. Maslov and his disciples (1992). By means of this approach linearizing convex problems, Hamilton-Jacobi equations with a convex Hamiltonian and their applications to problems of mathematical physics are studied.
Optimal control problems and differential games are connected one way or another with a search for solutions of Hamilton-Jacobi-Bellman, Isaaks equations. To solve such equations, constructive and numerical methods (including grid ones) were developed (Subbotin et al., 1993, 1994). An important result of the theory of minimax solutions of first-order PDE being a base for differential game theory is proving the equivalence of concepts of minimax and viscosity solutions (Subbotin, 1995).
Within the frameworks of minimax solution concept originating from the theory of position differential games (Krasovsky and Subbotin, 1988) developed by school of N. N. Krasovsky on the base of minimax evaluations and operations, theorems of existence and uniqueness, correctness and content-richness of minimax solution concept for various types of boundary problems of partial first-order PDE were proved.
Despite available theoretical results in this area, the issue of Hamilton-Jacobi-Isaacs equation solution in the problems of differential games with non-linear indefinite dynamic objects in the rate of their functioning persists and is important today.

2. NON-LINEAR OPTIMAL REGULATOR

2.1. Problem statement

Consider a dynamical non-linear uncertain system described by the ordinary differential equation

$$\frac{d}{dt}x(t) = f(x) + g_1(x)w(t) + g_2(x)u(t), \;\; x(t_0) = x_0, \qquad y(t) = Hx(t)\,. \tag{2.1}$$

Here $x(\cdot) = \{x(t) \in R^n, t \in [t_0, T]\}$ is a state vector of the system; $x(\cdot) \in \Omega_x$, $X_0 \in \Omega_x$ is a range of possible initial conditions of the system; $y \in R^m$, $m \le n$ is an output of the system; $u \in R^r$ is a control; $w \in R^k$ is a disturbance; $f(x)$, $g_1(x)$, $g_2(x)$ are continuous matrix functions.

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It is assumed that for all x system (2.1) is controllable and observable, $t \in R^{+}$. In addition, assume that functions $f(x),\ g_1(x),\ g_2(x)$ are smooth enough (C_∞), so that for any $(t_0, x_0) \in R_+ \times \Omega_x$ only one solution $x(t, t_0, x_0)$ of (2.1) equation is possible and the corresponding output of the system $y(t) = Cx(t, x_0)$ is unique.

Assumption 2.1. The vector function $f(x)$ is continuous differentiable with respect to $x \in \Omega_x$, i.e. $f(\cdot) \in C^1(\Omega_x)$ and $g_1(\cdot),\ g_2(\cdot) \in C^0(\Omega_x)$.

Assumption 2.2. Without loss of generality, assume that condition $x = 0 \in \Omega_x$ is a point of equilibrium of the system under $u = 0,\ w = 0$, so that $f(0) = 0$ and $g_1(x) \neq 0$, $g_2(x) \neq 0,\ \forall x \in \Omega_x$.

While considering disturbance $w(t)$ as an action of some player against successful performance of a control problem, we state the control problem in the sense of a differential game of two players: G_u and G_w. Controls $u(t) \in U$ and $w(t) \in W$ will be organized using the state feedback principle.

So, in the present section, the problem of control of nonlinear uncertain object (2.1) will be considered in the sense of the minimax theory.

Introduce the cost functional of the differential game

$$J(x,u,w) = K(x(T)) + \frac{1}{2}\int_{t_0}^{T}\left\{ y^{\mathrm{T}}(t) Q\, y(t) + u^{\mathrm{T}}(t) R\, u(t) - w^{\mathrm{T}}(t) P w(t) \right\} dt. \tag{2.2}$$

In functional (2.2) a symmetrical matrix Q is at least positively semidefinite, P and R matrices are positive definite.

Assumption 2.3. Limits on control actions U and W, where the task is executed successfully differential game, determined by the respective values of the matrices R, P, parameters $\sigma_i,\ i = 1,...k$ and matrix $g_1(x),\ g_2(x)$.

Let element $\xi = (x(t), u(t), w(t))$ be a permissible controllable process. Functions of class

$$x(\cdot) \in C^1\left([t_0, T], R^n\right),\ u(\cdot) \in C^1\left([t_0, T], R^r\right),$$
$$w(\cdot) \in C^1\left([t_0, T], R^k\right)$$

will be considered as permissible elements $\xi = (x(t), u(t), w(t))$.

The problem of differential game consists in construction of an optimal strategy with feedback for players G_u and G_w, i.e. in finding of control $u(t)$ minimizing a functional of (2.2) on the object (2.1) under corresponding counteraction to control $w(t)$.

2.2. Optimal controls of differential game

Make two assumptions:

Assumption 2.4. Let $f(x),\ g_1(x),\ g_2(x)$ be smooth enough functions, so that function $V(t,x)$ determined as

$$V(t,x) = \inf_{u \in U} \sup_{w \in W} J(x,u,w) \tag{2.3}$$

is a differentiable function under any permissible strategies of players $G_w,\ G_u \in L_2(0,\infty)$.

Assumption 2.5. A function $V(t,x)$ determined in (2.3) is locally Lipschitz in Ω_x.

In general case, value of an assigned function $V(t,x)$ is a solution of dynamic programming problem connected with partial differential equation of the first order (first order PDE) Hamilton-Jacobi-Isaacs (Issacs, 1965).

$$\begin{aligned} &\frac{\partial V(t,x)}{\partial t} + \min_u \max_w H\left\{ x, u, w, \frac{\partial V(t,x)}{\partial x} \right\} = 0, \\ &V(T, x(T)) = K(x(T)), \end{aligned} \tag{2.4}$$

where H is Hamiltonian

$$\begin{aligned} &H\left\{ x, u, w, \frac{\partial V(t,x)}{\partial x} \right\} = \\ &= \frac{1}{2}\left\{ y^{\mathrm{T}}(t) Q\, y(t) + u^{\mathrm{T}}(t) R\, u(t) - w^{\mathrm{T}}(t) P w(t) \right\} + \\ &+ \frac{\partial V(t,x)}{\partial x}\left\{ f(x) + g_1(x) w(t) + g_2(x) u(t) \right\}. \end{aligned} \tag{2.5}$$

Optimum controls $u(t)$ and $w(t)$ when performing Assumptions 2.3, defined by the relations

$$\begin{aligned} w(t) &= P^{-1} g_1^{\mathrm{T}}(x) \left\{ \frac{\partial V(t,x)}{\partial x} \right\}^{\mathrm{T}}, \\ u(t) &= -R^{-1} g_2^{\mathrm{T}}(x) \left\{ \frac{\partial V(t,x)}{\partial x} \right\}^{\mathrm{T}}, \end{aligned} \tag{2.6}$$

where vector $\partial V(x)/\partial x$ is determined by solution of Hamilton-Jacobi-Isaacs equation:

$$\begin{aligned} &\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x) - \\ &- \frac{1}{2}\frac{\partial V(t,x)}{\partial x} \Pi(x) \left\{ \frac{\partial V(t,x)}{\partial x} \right\}^{\mathrm{T}} + \frac{1}{2} x^{\mathrm{T}}(t) H^{\mathrm{T}} Q H x(t) = 0, \\ &V(T, x(T)) = K(x(T)), \end{aligned} \tag{2.7}$$

where

$$\Pi(x) = g_2(x) R^{-1} g_2^{\mathrm{T}}(x) - g_1(x) P^{-1} g_1^{\mathrm{T}}(x). \tag{2.8}$$

The main difficulty under implementation of controls in form (2.6) consists in finding of vector $\partial V(x)/\partial x(t)$ satisfying scalar partial derivative equation (2.7).

2.3. Conditions of existence of optimal solution

Conditions of existence of optimal solution of the set problem are determined by properties of matrix $\Pi(x)$. To determine properties of this matrix, consider in this section problem of synthesis of stabilizing controls for system (2.1), i.e. consider the problem with unlimited time of transition process. Quality functional for such a problem has the form

$$J(x,u,w) = = \lim_{T \to \infty} \frac{1}{2}\int_0^T \left\{ y^{\mathrm{T}}(t) Q\, y(t) + u^{\mathrm{T}}(t) R\, u(t) - w^{\mathrm{T}}(t) P w(t) \right\} dt. \tag{2.9}$$