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Stability Investigation of Difference Schemes for Gas Dynamics Equations ⋆

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Abstract: Problems of effective control of gas flow are closely related to the system of equations describing the motion of the gas in the gas collection network of a gas field. Due to the great length of gas gathering networks and the necessity to consider the compressibility of the gas, pressure and flow distributions are described by the Navier - Stokes equations. For the successful application of difference methods for solving these equations, their stability must be thoroughly investigated. This is the goal of current article.

Keywords: stability, difference schemes, gas, Navier - Stokes equations

INTRODUCTION

The effective gas flow control problem is closely connected to solving differential equation system which models gas movement in gas collecting system pipes. Due to vast length of gas collecting system and necessity to take gas compressibility into account, pressure and flow rate are modeled by Navier-Stokes equations (see Korotaev (1989)). To successfully apply finite-difference methods in solving these equations, it is necessary to investigate their stability. This investigation can be carried out using spectral condition and Babenko-Gelfand method (see Tsynkov (2006))

1. PROBLEM FORMULATION

Let us consider a linear segment of a gas network. To describe pressure and flow distributions, we'll use the following equations

$$\frac{\partial Q(x,t)}{\partial t} = -\frac{f}{\rho}\frac{\partial P(x,t)}{\partial x} - \frac{8\pi\mu}{f\rho}Q(x,t), \tag{1}$$

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\rho}{f}c^2\frac{\partial Q(x,t)}{\partial x} \tag{2}$$

(1) and (2) follow from momentum and mass conservation laws respectively (see Korotaev (1989)). The following notations are used: $x \in [0, L]$ and $t \in [0, T]$ are space and time coordinates, $Q(x,t)$ and $P(x,t)$ — flow and pressure functions, $\rho(P)$ and μ – gas density and dynamic viscosity, f – pipe cross-section area, $c = \sqrt{E/\rho}$ – speed of sound in the gas flow, E – elastic modulus. As boundary and initial conditions, let us assume

$$\begin{array}{c} P(x,0) = P^0(x), Q(x,0) = Q^0(x), \\ P(0,t) = 0, P(L,t) = 0. \end{array} \tag{3}$$

Equations (1) - (2) are a nonlinear partial differential equations system. With constant ρ and c (f is assumed fixed) these equations become linear with constant coefficients. This property is used in finite difference stability investigation.

Let us consider (1) - (3) and the corresponding finite difference equations. Let the grid be defined by lines $x = mh$, $t = n\tau$, where $m = \overline{0, M}$, $h = L/M$, $n = \overline{0, T/\tau}$, $\tau = rh$, $r = const$. Unknown grid functions are designated P_m^n and Q_m^n, so boundary conditions (3) are transformed to

$$\begin{array}{c} P_m^0 = \varphi_{P_m}, Q_m^0 = \varphi_{Q_m}, m = \overline{0, M}, M = [L/h] \\ P_0^{n+1} = 0, P_M^{n+1} = 0, n = \overline{0, [T/\tau]}. \end{array} \tag{4}$$

Thus, to investigate the stability of the difference scheme approximating (1) - (3), let us use spectral analysis and frozen coefficient principle (Babenko and Gelfand condition, see Tsynkov (2006)). We'll consider 3 subproblems with boundary conditions (4):

Proposition 1. In the initial problem, we'll assume $\rho(P)$, $c(P)$ satisfy the following constraints:

$$\rho_{min} \le \rho(P) \le \rho_{max}, c_{min} \le c(P) \le c_{max}, \tag{5}$$

where $\rho_{min} = \rho(0)$, $\rho_{max} = \rho(P_{max})$ are obtained from Benedict–Webb–Rubin equation (see Korotaev (1989)). Coefficients are to be fixed in intervals $[0, \rho_{max}]$, $[0, c_{max}]$, and the difference equations will be considered not only in

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$0 \le m \le M$, but with any positive m, i.e. $m = 0, \pm 1, \dots$ for all $n = 0, 1, \dots, [T/\tau] - 1$.

Proposition 2. If the coefficients $\rho(P)$ and $c(P)$ and frozen at the left end of the interval $[0, L]$, i.e. at the points $P = 0$, $\tilde{t}$, $c_1 = c(P_0^n)$, $\rho_1 = \rho(P_0^n)$, $m = \overline{1, M}$, $P_0^n = 0$, $0 \le \tilde{t} \le [T/\tau] = M$, it is natural to consider only the grid functions of the form $P_m^n \to 0$, $Q_m^n \to 0$ at $m \to +\infty$.

Proposition 3. If the coefficients $\rho(P)$ and $c(P)$ and frozen at the right end of the interval $[0, L]$, let us assume $m = \dots, -2, -1, 0, 1, 2, \dots, M-1, P_M^{n+1} = 0$. Then we only consider grid functions that satisfy $P_m^n \to 0$, $Q_m^n \to 0$ at $m \to -\infty$.

For each of these 3 subproblems, we search for eigenvalues of the operator $A : P^n, Q^n \to P^{n+1}, Q^{n+1}$ which admit solutions $P_m^n = \lambda^n P_m^0$, $Q_m^n = \lambda^n Q_m^0$. In the case of problem 1, functions P_m^n, Q_m^n, $m = 0, \pm 1, \dots$ must be bounded, in the case of problem 2 $P_m^n \to 0$, $Q_m^n \to 0$, $m \to +\infty$, and in the case of problem 3 $P_m^n \to 0$, $Q_m^n \to 0$, $m \to -\infty$.

For the difference scheme to be stable, all of the eigenvalues of each of 3 subproblems must be located inside the unit circle $|\lambda| \le 1$ (see Tsynkov (2006)).

2. EXPLICIT SCHEMES

Consider the stability of the explicit scheme approximation

$$\begin{aligned} P_m^{n+1} - P_m^n &= \frac{\tau}{h}\left(-\frac{\rho}{f}c^2\right)_{m}^{n}(Q_{m+1}^n - Q_m^n), \\ Q_m^{n+1} - Q_m^n &= \frac{\tau}{h}\left(-\frac{f}{\rho}\right)_m^n (P_{m+1}^n - P_m^n) - \\ &- \left(\frac{8\pi\mu}{f\rho}\right)_m^n Q_m^n \tau, \end{aligned} \tag{6}$$

or

$$\begin{aligned} P_m^{n+1} &= P_m^n - \beta_m^n(Q_{m+1}^n - Q_m^n), \beta_m^n = \left(\rho_m c_m^2/f\right)^n r, \\ Q_m^{n+1} &= Q_m^n - S_{1m}^n(P_{m+1}^n - P_m^n) - S_{2m}^n Q_m^n, \\ S_{1m}^n &= (f/\rho_m)^n r, S_{2m}^n = (8\pi\mu/f\rho_m)^n \tau. \end{aligned} \tag{7}$$

According to reasoning of section 1, let us consider the solution of problem 1 in form

$$P_m^n = \lambda^n P^0 e^{i\alpha m}, Q_m^n = \lambda^n Q^0 e^{i\alpha m} \tag{8}$$

where $m = 0, \pm 1, \dots, \alpha$ - real parameter. From (6)-(8) follows that

$$\begin{aligned} P^0(\lambda - 1) + Q^0 \widehat{\beta}_m (e^{i\alpha} - 1) = 0, \\ P^0(\widehat{S}_{1m}(e^{i\alpha} - 1)) + Q^0(\lambda - 1 + (\widehat{S}_{2m}) = 0, \end{aligned} \tag{9}$$

where $\widehat{\beta}, \widehat{S}_1, \widehat{S}_2$ numbers from (7) defined by arbitrary points on intervals (5). System (9) has a nontrivial solution under the condition of equality to zero of its determinant

$$(\lambda - 1)(\lambda - 1 + \widehat{S}_{2m}) - \widehat{S}_{1m}\widehat{\beta}_m (e^{i\alpha} - 1)^2 = 0. \tag{10}$$

Since $\widehat{S}_2$ has the order τ, let $\widehat{S}_2 = 0$. From (10) follows

$$\lambda_1 = 1 + \sqrt{\widehat{S}_1 \widehat{\beta}}(e^{i\alpha} - 1), \lambda_2 = 1 - \sqrt{\widehat{S}_1 \widehat{\beta}}(e^{i\alpha} - 1). \tag{11}$$

It is necessary to evaluate the module of the complex values λ_1 and λ_2. First of all let us evaluate the module of the expression $d + be^{i\alpha}$, where d and b - real numbers. It can be seen that

$$|d| - |b| \le |d + be^{i\alpha}| \le |d| + |b| \tag{12}$$

Consider the root λ_2:

$$\begin{aligned} d = 1 + \sqrt{\widehat{S}_1 \widehat{\beta}}, b = -\sqrt{\widehat{S}_1 \widehat{\beta}}, \\ |\lambda_2| \le |d| + |b| = 1 + 2\sqrt{\widehat{S}_1 \widehat{\beta}} = 1 + 2\widehat{c}r. \end{aligned} \tag{13}$$

Thus, the eigenvalues of the transition from P_m^n, Q_m^n to P_m^{n+1}, Q_m^{n+1} 1 in the problem 1 may be greater than one. Due to the analytic dependence of the roots λ_1 and λ_2 from small parameter $\widehat{S}_2$, inequality (12) holds if $\widehat{S}_2$ sufficiently small. Thus, the explicit scheme (6) will be unstable.

2.1 Problem 1

Considered difference scheme approximates (1) - (3) with the first order of smallness relative to h. To construct a second order difference scheme, we use the approach described in Tsynkov (2006), that is based on the method of undetermined coefficients.

Let us introduce the difference function with undetermined coefficients

$$\begin{aligned} \Lambda P^n \equiv a^0 P_m^{n+1} + a_0 P_m^n + a_1 P_{m+1}^n + a_{-1} P_{m-1}^n + \\ + b_0 Q_m^n + b_1 Q_{m+1}^n + b_{-1} Q_{m-1}^n, \\ \Lambda Q^n \equiv c_0 P_m^n + c_1 P_{m+1}^n + c_{-1} P_{m-1}^n + d^0 Q_m^{n+1} + \\ + d_0 Q_m^n + d_1 Q_{m+1}^n + d_{-1} Q_{m-1}^n. \end{aligned} \tag{14}$$

Let us introduce the function ΛP, ΛQ, using the equation (1) and (2)

$$\Lambda P = P_t + \bar{\beta} Q_x, \Lambda Q = Q_t + \bar{S}_1 P_x + \bar{S}_2 Q. \tag{15}$$

Hence

$$\begin{aligned} P_t = \Lambda P - \bar{\beta} Q_x, Q_t = \Lambda Q - \bar{S}_1 P_x - \bar{S}_2 Q, \\ P_{tt} = (\Lambda P)_t - \bar{\beta}(Q_t)_x = (\Lambda P)_t - \bar{\beta}(\Lambda Q)_x + \\ + \bar{\beta}\bar{S}_1 P_{xx} + \bar{\beta}\bar{S}_2 Q_x, \\ Q_{tt} = (\Lambda Q)_t - \bar{S}_1(\Lambda P - \bar{\beta} Q_x)_x - \bar{S}_2 Q_t = \\ = (\Lambda Q)_t - \bar{S}_1(\Lambda P)_x + \bar{\beta}\bar{S}_1 Q_{xx} - \bar{S}_2 \Lambda Q + \\ + \bar{S}_1 \bar{S}_2 P_x + \bar{S}_2^2 Q. \end{aligned} \tag{16}$$

Here $\bar{\beta} = \rho c^2/f$, $\bar{S}_1 = f/\rho$, $\bar{S}_2 = 8\pi\mu/f\rho$ for arbitrary fixed values ρ, c, S_1, S_2 from (5). Using Taylor's formula and equality (14) - (16), we can write a function ΛP^n and ΛQ^n in the form

$$\begin{aligned} \Lambda P_h = a^0 P_m^n + a^0 rh(P_t)_m^n + a^0 r^2 h^2 (P_{tt})_m^n/2 + \\ + a_0 P_m^n + a_1 P_m^n + a_{-1} P_m^n + (a_1 - a_{-1})h(P_x)_m^n + \\ + (a_1 + a_{-1})h^2/2(P_{xx})_m^n + b_0 Q_m^n + b_1 Q_m^n + \\ + b_{-1} Q_m^n + h(b_1 - b_{-1})(Q_x)_m^n + \\ + + h^2(b_1 + b_{-1})(Q_{xx})_m^n/2 = \\ = a^0 rh(\Lambda P)_m^n + a^0 r^2 h^2 (\Lambda P_t)_m^n/2 - \\ - \bar{\beta} a^0 r^2 h^2 (\Lambda Q_x)_m^n/2 + \\ + P_m^n(a^0 + a_0 + a_1 + a_{-1}) + (P_x)_m^n h(a_1 - a_{-1}) + \\ + (P_{xx})_m^n (a^0 r^2 h^2 \bar{\beta}\bar{S}_1/2 + \\ + (a_1 + a_{-1})h^2/2) + Q_m^n(b_0 + b_1 + b_{-1}) + \\ -(Q_x)_m^n(h(b_1 - b_{-1}) - -a^0 rh\bar{\beta} + \\ + a^0 r^2 h^2 \bar{\beta}\bar{S}_2/2) + (Q_{xx})_m^n(b_1 + b_{-1})h^2/2 + o(h^2). \end{aligned} \tag{17}$$