Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-10 (2016) 001–006

Delayed Resource Allocation Optimization with Applications in Population Dynamics ⋆

Erik I. Verriest * **Gunther Dirr** ** **Uwe Helmke** ***

* *School of ECE, Georgia Institute of Technology, Atlanta, GA,USA (email: erik.verriest@ece.gatech.edu).*
** *Institute of Mathematics, University of Würzburg, 97074 Würzburg,Germany (email: dirr@mathematik.uni-wuerzburg.de)*
*** *Institute of Mathematics, University of Würzburg, 97074 Würzburg,Germany (email: helmke@mathematik.uni-wuerzburg.de)*

Abstract: The contribution of this paper is twofold. First a simplified derivation of the maximum principle for optimal control problems with delay is presented based on first principles. Then the result is applied to an investigation into the dynamic optimal resource allocation problem in population dynamics involving delayed action.

Keywords: delay system, optimal control resource allocation

1. INTRODUCTION

This paper considers a resource allocation problem for two populations that can be thought of as predator and prey, or consumer and food resource, as in sheep and grassland (Noy-Meir (1978),Woodward and Wake (1994)). A similar model also applies to fisheries (Chakraborty et al. (2013)). We will denote the populations simply by R for the resource, and S (for Sheep or Species) for the consumers. In the absence of a delay, we assume the following base model.

1. The resource shows a natural depletion or decay with rate μ (the natural death rate of prey) and a growth rate c if fully invested, i.e., if all grass seeds go to replenishing the field.
2. The consumer population, S, has a natural death rate ν which is offset by a growth proportional to the resource allocated.

Hence the model leads to the bilinear system

$$\dot{R}(t) = -\mu R(t) + cR(t)u(t) \tag{1}$$

$$\dot{S}(t) = -\nu S(t) + \omega R(t)(1 - u(t)), \tag{2}$$

where the control $0 \leq u(t) \leq 1$ is the allocation of the resource. The parameters ω and c take different efficiencies of the allocated resource into account. It is desired to maximize S either at a fixed or a free final time. Note that this model differs from the traditional predator-prey model which has a mass action form in the S equation.
More generally, one may consider multiple population models with several resources of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_r(t) \\ x_s(t) \end{bmatrix} = F(x_r(t), x_s(t)) + [C \circ U(t)]x_r(t) \tag{3}$$

where $\dim x_r = n_r$, $\dim x_s = n_s$ and where $U \in [0,1]^{(n_r+n_s)\times n_r}$ is a resource allocation matrix with entries u_{ij}, the allocation of resource x_{r_j} for the growth of x_i. C is the efficiency matrix C_{ij} indicating how a unit allocation of x_j contributes to the rate of x_i. The operation $\circ$ is the Hadamard product, defined by $(C \circ U)_{ij} = C_{ij}U_{ij}$. The matrix U is normalized by $\sum_i u_{ij} = 1$. Thus, if $\mathbf{1} = \mathrm{col}(1, \ldots, 1)$ of appropriate dimension, then the control matrix satisfies

$$\mathbf{1}^\top U = \mathbf{1}^\top.$$

The above models are only valid if there are no delays from allocation to consumption. Typically, however, such an assumption is not realistic. For simplicity we consider the case of a single delay. This changes the model to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_r(t) \\ x_s(t) \end{bmatrix} = F(x_r(t), x_s(t)) + [C \circ U(t-\tau)]x_r(t-\tau). \tag{4}$$

In Section 2, we derive necessary conditions for the general optimal allocation problem (4), but with a nondelayed representation, $v(t) = \mathrm{vec}\, U(t-\tau)$, for the vector of all controls.

$$\dot{x}(t) = f(x(t), x(t - \tau(t)), v(t)).$$

The objective here is to maximize some function of the population at a fixed or free terminal time and augmented with an integral over time. Thus let $\Phi(x(T), T) + \int_0^T L(x(t), v(t))\, dt$ be this performance index. Necessary conditions for optimality and the maximum principle for systems with delay have been presented earlier, see for instance Halanay (1968),Bien and Chyung (1980), Patel et al. (1982), Bokov (2010), and Malek-Zavarei and Jamshidi (1987). However, the analysis in these references is either very lengthy or restrictive with respect to the model class or the admissible controls. We present a more streamlined derivation based on *first principles*, which can be made rigorous easily. We believe that such a derivation

⋆ E.I. Verriest gratefully acknowledges the hospitality at the Center of Interdisciplinary Mathematical Research (IFZM) of the Julius-Maximilians-University Würzburg during his Professional Development Leave from Georgia Tech during the Fall of 2015.
The second and third author, G. Dirr and U. Helmke, have been supported by the grants HE 1858/13-1 and HE 1858/14-1 from the German Research Foundation (DFG).
It is with deep sorrow that we report the demise of our friend and co-author Prof. Uwe Helmke while preparing the final manuscript.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.07.463

is also more pedagogical and illuminating than the standard approach where the adjoint equations enter ab initio in a more obscure way. Full details will appear elsewhere. In Section 3, we consider in detail a simple but relevant model for social insect population dynamics and sheep farming. This work adds to our past work on delay models in ecology and epidemiology (Verriest and Pepe (2007), Briat and Verriest (2009)).

2. OPTIMAL CONTROL FOR SMOOTH SYSTEMS WITH DELAY

Let

$$\dot{x}(t) = f(x(t), x(t-\tau), v(t)),$$

with initial data $x(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$ given and the terminal state $x(T)$ unspecified. Let the objective function, to be maximized, be

$$J = \Phi(x(T), T) + \int_0^T L(x(t), v(t))\,\mathrm{d}t,$$

whose final time $T \geq 0$ may be specified or unspecified. For greater generality, we assume the latter.

The simple perturbation approach, assuming that $f(x, y, v)$, $L(x, v)$, and $\Phi(x, t)$ are smooth functions of x, y, v and t, is only applicable when the control is not constrained. The derivation that follows needs to allow for large control perturbations and cannot rely on simple first order perturbation terms with respect to the control.

Suppose the optimal performance J_0 is given by an optimal control v_0, which induces the state x_0, and an optimal terminal time $T_0 \geq 0$.

By definition of global optimality, if v is *any* other admissible control function, and T *any* other terminal time, then the induced state x must result in an inferior (or equal) performance. Thus $J_0 - J \geq 0$ or, for all admissible controls v, and for all T,

$$\Phi(x_0(T_0), T_0) + \int_0^{T_0} L(x_0(t), v_0(t))\,\mathrm{d}t - \Phi(x(T), T) - \int_0^T L(x(t), v(t))\,\mathrm{d}t \geq 0. \tag{5}$$

But x_0 and x respectively satisfy

$$\dot{x}_0(t) = f(x_0(t), x_0(t-\tau), v_0(t)) \tag{6}$$

$$\dot{x}(t) = f(x(t), x(t-\tau), v(t)) \tag{7}$$

with the same initial data, ϕ. Hence, the left hand side of (5) will not change if $\lambda^\top(t)[f(x_0(t), x_0(t-\tau), v_0(t)) - \dot{x}_0(t)]$ is added to the integrand in the first integral, where λ is for now a completely arbitrary function. Likewise we add $\lambda^\top(t)[f(x(t), x(t-\tau), v(t)) - \dot{x}(t)]$ to the integrand of the second integral, noting that the *same* Lagrange multiplier $\lambda(t)$ is *chosen*. Hence the necessary condition (5) may be replaced by

$$\Phi(x_0(T_0), T_0) - \Phi(x(T), T) + \int_0^{T_0} L(x_0(t), v_0(t)) + \lambda^\top(t)(f(x_0(t), x_0(t-\tau), v_0(t)) - \dot{x}_0(t))\,\mathrm{d}t - \int_0^T L(x(t), v(t)) + \lambda^\top(t)(f(x(t), x(t-\tau), v(t)) - \dot{x}(t))\,\mathrm{d}t \geq 0.$$

Restrict now the function λ to be differentiable. Integrating by parts yields the equivalent condition

$$\Phi(x_0(T_0), T_0) - \Phi(x(T), T) - \lambda^\top(T_0)x_0(T_0) + \lambda^\top(T)x(T) + \int_0^{T_0} L(x_0(t), v_0(t)) + \lambda^\top(t)f(x_0(t), x_0(t-\tau), v_0(t)) + \dot{\lambda}^\top(t)x_0(t)\,\mathrm{d}t - \int_0^{T_0} L(x(t), v(t)) + \lambda^\top(t)f(x(t), x(t-\tau), v(t)) + \dot{\lambda}^\top(t)x(t)\,\mathrm{d}t \geq 0.$$

Defining the Hamiltonian associated with this problem by

$$H(x(t), x(t-\tau), v(t), \lambda(t)) := L(x(t), v(t)) + \lambda^\top(t)f(x(t), x(t-\tau), v(t)), \tag{8}$$

the necessary condition for optimality states then that for all T and for all admissible controls $v(\cdot)$ defined on $(0, \max(T, T_0))$ it holds that

$$\Phi(x_0(T_0), T_0) - \Phi(x(T), T) - \lambda^\top(T_0)x_0(T_0) + \lambda^\top(T)x(T) + \int_0^{T_0} [H(x_0(t), x_0(t-\tau), v_0(t), \lambda(t)) + \dot{\lambda}^\top(t)x_0(t)]\,\mathrm{d}t - \int_0^T [H(x(t), x(t-\tau), v(t), \lambda(t)) + \dot{\lambda}^\top(t)x(t)]\,\mathrm{d}t \geq 0.$$

In particular, for T in a neighborhood of T_0, we set $T = T_0 + \epsilon$, and for sufficiently small ϵ, we obtain by a first order Riemann sum approximation

$$\Phi(x_0(T_0), T_0) - \Phi(x(T), T) - \lambda^\top(T_0)x_0(T_0) + \lambda^\top(T)x(T) + \int_0^{T_0} [H(x_0(t), x_0(t-\tau), v_0(t), \lambda(t)) + - H(x(t), x(t-\tau), v(t), \lambda(t)) + \dot{\lambda}^\top(t)(x_0(t) - x(t))]\,\mathrm{d}t - \epsilon[H(x(T_0), x(T_0-\tau), v(T_0), \lambda(T_0)) + \dot{\lambda}^\top(T_0)x(T_0))] \geq 0.$$

For simplicity of notation, let $y_0(t)$ and $y(t)$ respectively represent $x_0(t-\tau)$ and $x(t-\tau)$. The integrand of the above integral term is then (suppressing the time argument)

$$I := H(x_0, y_0, v_0, \lambda) - H(x, y, v, \lambda) + \dot{\lambda}^\top(x_0 - x).$$

which may be transformed by adding and subtracting $H(x, y, v_0, \lambda)$,

$$I = [H(x_0, y_0, v_0, \lambda) - H(x, y, v_0, \lambda) + \dot{\lambda}^\top(x_0 - x)] + [H(x, y, v_0, \lambda) - H(x, y, v, \lambda)],$$

into an expression involving v_0 and one involving fixed x and y. Likewise the non-integral terms are rewritten as

$$\begin{aligned}&\Phi(x_0(T_0), T_0) - \Phi(x(T), T) - \lambda^\top(T_0)x_0(T_0) + \lambda^\top(T)x(T)\\ &= [\Phi(x_0(T_0), T_0) - \Phi(x(T_0), T_0) - \lambda^\top(T_0)(x_0(T_0) - x(T_0))]\\ &+ [\Phi(x(T_0), T_0) - \Phi(x(T), T) + \lambda^\top(T)x(T) - \lambda^\top(T_0)x(T_0)].\end{aligned}$$

These steps are the key to derive the maximum principle. Since $T = T_0 + \epsilon$, and in view of the differentiability of x and λ, this yields up to first order in ϵ the approximation

$$\Phi(x_0(T_0), T_0) - \Phi(x(T_0), T_0) - \lambda^\top(T_0)(x_0(T_0) - x(T_0)) - \epsilon\left[\frac{\partial \Phi(x(T_0), T_0)}{\partial x}\dot{x}(T_0) + \frac{\partial \Phi(x(T_0), T_0)}{\partial T} + - \dot{\lambda}^\top(T_0)x(T_0) - \lambda^\top(T_0)\dot{x}(T_0))\right].$$