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Fundamental Bounds for Stabilizability of Continuous-Time Systems under Stochastic Multiplicative Uncertainty and Delay ⋆

Tian Qi * **Jie Chen** **

* *School of Automation Science and Engineering, South China University of Technology, Guangzhou, China (e-mail: auqt@scut.edu.cn).*

** *Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong (e-mail: jichen@cityu.edu.hk)*

Abstract: This paper is devoted to stabilization problem of linear time-invariant (LTI) continuous-time systems under stochastic multiplicalive uncertainty and time-delay. In its full generality, we model the stochastic multiplicative uncertainty as a random process with a certain distribution. We assess the stability of system based on mean-square criteria. Our main contribution includes the fundamental condition, both necessary and sufficient, which insures that the single-input single-output systems (SISO) can be mean-square stabilized by output feedback. This condition provides explicitly a fundamental limit on mean-square staibilizability imposed by the signal-to-noise ratio (SNR), the system's unstable poles, nonminimum phase zeros and time-delay.

Keywords: Mean square stability and stabilization, constant time-delay, stochastic multiplicative uncertainty.

1. INTRODUCTION

During last four decades, considerable researchers have denoted their attention to control problems of linear time-invariant (LTI) systems under the effect of stochastic multiplicative noise, and many significant results are available in the literature (see, e.g., Hinrichsen (1996), Lu (2000), Lu (2002), Willems (1971)). Apart from the longstanding interest in this class of systems by themselves, recent development in networked control systems shows the relevance of stochastic multiplicative noises to uncertainties in communication channels, including packet loss (Elia (2005), Elia (2011), Sinopoli (2004), Wu (2007), Xiao (2012)), quantization error (Qiu (2013), Qi (2008), Su (2011)) and channel fading (Braslavsky (2007), Elia (2005), Freudenberg (2011)). Thus, the study of control under stochastic multiplicative uncertainties has prevailed in the recent control work (see, e.g., Nair (2007), Special Issue (2004), Special Issue (2007), Braslavsky (2007), Qiu (2013), You (2011) and the references therein).

Another sustained issue in control theory is time-delay, which occurs often in systems in engineering, biology, physics, et al. In general, delay can lead to degraded performance, poor robustness and even instability of systems. Contemporary studies reveal such conspicuous negative effects in networked control systems. Due to its intrinsic significance and renewed interest, the study of time-delay systems has made remarkable progress, and various approaches have been available for stability analysis in time- and frequency-domain. (see, e.g., Niculescu (2001), Boukas (2002), Richard (2003), Gu (2003), Niculescu (2004)). Despite the developments in stability analysis, however, the stabilization of time-delay systems proves fundamentally more difficult and remains to be a daunting task.

In this paper we study the stabilization problem for LTI continuous-time SISO systems subject to stochastic multiplicative uncertainty and time-delay. Without loss of generality, we model the stochastic multiplicative uncertainty as a random process with a certain distribution. Under this formulation, the uncertainty can be interpreted as classes of erasure channels (Elia (2005)), a memoryless noisy communication channel with packet drop and random delay (Sinopoli (2004)), in the networked control setting. We assess the system's stability based on *mean-square* criteria from an input-output perspective; particularly, the output will be bounded statistically for every input to the system that is bounded. Our approach is inspired by the pioneering work of Willems and Blankenship (Willems (1971)) on a stochastic multiplicative white noise, who studied the closed-loop stability of SISO systems and obtained a necessary and sufficient condition for mean-square stability. This development shares much in common with robust stability analysis and leads to stability result reminiscent of small gain condition, herein dubbed as *mean-square small gain theorem*.

We employ the mean-square small gain approach to tackle our stabilization problem. While the constant time-delay factor is unreasonable in frequency-domain, the mean-square stabilization problem of SISO systems subject to time-delay generally requires solving an infinite-

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dimensional $\mathcal{H}_2$ optimal problem. Drawing upon rational approximation techniques, we solve this problem and provide an explicit stabilizability condition, both necessary and sufficient, for a system to be stabilized by output feedback in the mean-square sense. This condition characterizes the minimal requirement for mean-square stabilization problem in the SISO case in terms of the stochastic uncertainty SNR, the plant's unstable poles, nonminimum phase zeros, as well as time-delay. More special cases are further investigated. A stringent restriction is showed to be imposed on stochastic uncertainty SNR by increasing time-delay. Without regard to time-delay and nonminimum phase effect, the result is reminiscent of prior work on stabilization subject to channel SNR constraints (Li (2009a),Li (2009b),Rojas (2005),Rojas (2006),Braslavsky (2007)), and limited mean-square capacity (Xiao (2010)).

The notation used in this note is collected in the following. We denote the complex plane by $\mathbb{C}$. Let the open right half plane be denoted by $\mathbb{C}_+ := \{s : \mathrm{Re}(s) > 0\}$, the open left half plane by $\mathbb{C}_- := \{s : \mathrm{Re}(s) < 0\}$, and the imaginary axis by $\mathbb{C}_0$. For any complex number z, we denote by $\bar{z}$ and $\mathrm{Re}(z)$ its conjugate and real part. For any matrix $M \in \mathbb{C}^{m\times n}$, the conjugate transpose is denoted by M^H. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm. With respect to the imaginary axis $\mathbb{C}_0$, we shall frequently encounter the Hilbert space

$$\mathcal{L}_2 := \left\{ f : f(s) \text{ measurable in } \mathbb{C}_0, \right.$$
$$\left. \|f\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(j\omega)\|^2 d\omega < \infty \right\},$$

in which the inner product is defined as

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} f^H(j\omega) g(j\omega) d\omega.$$

It is well know that $\mathcal{L}_2$ can be decomposed into two orthogonal subspaces $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$:

$$\mathcal{H}_2 := \left\{ f : f(s) \text{ analytic in } \mathbb{C}_+, \right.$$
$$\left. \|f\|_2^2 = \sup_{\sigma>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(\sigma + j\omega)\|^2 d\omega < \infty \right\},$$

and

$$\mathcal{H}_2^\perp := \left\{ f : f(s) \text{ analytic in } \mathbb{C}_-, \right.$$
$$\left. \|f\|_2^2 = \sup_{\sigma<0} \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(\sigma + j\omega)\|^2 d\omega < \infty \right\}.$$

For any $f \in \mathcal{H}_2^\perp$ and $g \in \mathcal{H}_2$, $\langle f, g\rangle = 0$. Note that we use the same notation $\|\cdot\|_2$ for the spaces $\mathcal{L}_2$, $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$, but the the distinction will be self-evident from the context. Finally, we denote by $\mathcal{RH}_\infty$ the class of all stable, proper rational transfer function matrices.

2. PROBLEM FORMULATION

We focus on the uncertain systems depicted in Fig. 1, in which $P_\tau(s)$ represents a family of plants subject to a constant time-delay τ, with $P_0(s)$ being delay-free plant:

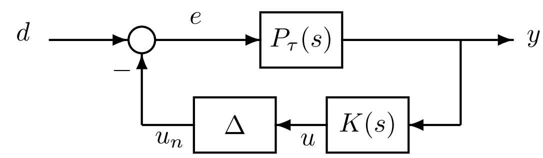

Fig. 1. Feedback system under multiplicative uncertainty and time-delay

$$P_\tau(s) = e^{-\tau s} P_0(s).$$

$K(s)$ denotes a LTI controller. The control signal u is interrupted by a static component Δ, such that

$$u_n(t) = \Delta(t)u(t),$$

where $\{\Delta(t)\}$ denotes a random variable with

$$\begin{aligned} E\{\Delta(t)\} &= \mu \neq 0, \\ E\{(\Delta(t) - \mu)^2\} &= \sigma^2 < \infty, \end{aligned} \tag{1}$$

for all $t > 0$. The input $d(t)$ is assumed to be independent of $\Delta(t)$.

This uncertainty can be applied to model uncertainties of communication channels. In Elia (2005), Elia shows that this uncertainty provides a rather general description for modeling erasure and possibly fading communication channels. Independent lossy, memoryless channels with i.i.d. stochastic processes, such as those with packet drops and random delays modeled by Bernoulli processes (Elia (2005), Sinopoli (2004)), can be also described using this uncertainty description.

We focus on the stabilization problem of systems in mean-square sense. The following mean-square stability definition is putted forward from an input-output perspective (Willems (1971)).

Definition 1. Let T be a linear system and Δ be a static white noise process with zero mean. Then the system in Fig. 2 is said to be mean-square input-output stable if every input process $\{d(t)\}$ with bounded variance $E\{d^2(t)\} < \infty$, generates well-defined error and output processes $\{e(t)\}$, $\{y(t)\}$, whose variances are also bounded, i.e., $E\{e^2(t)\} < \infty$ and $E\{y^2(t)\} < \infty$.

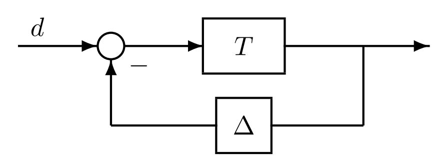

Fig. 2. Linear systems with stochastic multiplicative uncertainty

The following result, herein referred to as the mean-square small gain theorem, is adapted from Willems (1971), which provides a necessary and sufficient condition for the mean-square input-output stability. This result will play a pivotal role in our subsequent development.

Lemma 1. (Mean-Square Small Gain Theorem) Let T be a stable LTI system, and $\Delta(t)$ be a static white noise process with variance δ^2. Then, the system in Fig. 2 is mean-square input-output stable if and only if

$$\delta^2 \|T(s)\|_2^2 < 1. \tag{2}$$