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Delays. Nonlinearity. Synchronization

Daniela Danciu, Vladimir Răsvan

Department of Automation and Electronics, University of Craiova, A.I.Cuza, 13, Craiova, RO-200585, Romania (e-mail: {ddanciu, vrasvan}@automation.ucv.ro)

Abstract: A rather common model of the synchronization is represented by several local oscillators coupled to some (possibly distributed) transmission environment. This model goes back to Huygens. If the transmission environment is represented by a one dimensional distributed parameter structure (vibrating string in the mechanical case or an electric transmission line for electrical systems) then some functional differential equations may be associated by integration along the characteristics. Consequently, the synchronization of the two local oscillators can be analyzed as a problem of forced oscillations for functional differential equations. In this paper two LC local nonlinear oscillators are viewed as connected to a lossless LC transmission line of infinite length. This is the electrical analogue of the Huygens like case where two nonlinear pendula are coupled to a vibrating string.

Keywords: Synchronization, Nonlinear oscillators, LC transmission line, Popov criterion, Time delay

1. INTRODUCTION AND PROBLEM STATEMENT

A. In the contemporary research on systems' dynamics synchronization plays a distinct role. Its importance is emphasized in a more or less recent monograph Pikovsky et al. (2001) where it is considered a universal concept in nonlinear science (as its title states). It is not our purpose to elaborate on the concept but just to indicate its place within the qualitative study of dynamics. For instance, the standard example of Huygens is concerned with two pendula hanging on a wall or on a system of ropes that connect then thus "contributing" to their synchronization (observed by Huygens himself). The theoretical generalization of this old classical model is to consider oscillations in the sense of Lagrange i.e. any dynamical system described by ordinary differential equations (because Lagrange used to call oscillation - any transient i.e. non-steady state evolution of a system regardless it displays or not a periodic orbit). As a consequence there were considered oscillations coupled through diffusive or lattice structures Hale (1994, 1997). In the aforementioned papers the oscillators had the property of displaying a unique stable limit cycle accounting for a unique stable self-sustained periodic oscillation. Under these circumstances the frequencies and the phases of the oscillations may be viewed as restricted to a torus and smaller the torus dimension is – stronger is the synchronization Ermentrout and Koppell (1984); Koppell and Ermentrout (1986).

Another source for building synchronization models is to consider the coupling through the so called *complex interactions* among which one can mention the nonlinear couplings – the case of combustion modeling due to Kuramoto and Frank-Kamenetskii, the Power Grids case or the morphogenetic model of A. M. Turing – but also the linear distributed couplings – with diffusion or with propagation Hale (2004); Lepri and Pikovsky (2014). The linear couplings with delays may be also considered here Earl and Strogatz (2003); Fox et al. (2001).

B. In this research we shall consider the synchronization problem suggested by Lepri and Pikovsky (2014), at its turn generated by the classical Huygens synchronization problem. In the aforementioned paper, two nonlinear pendula without damping are hanging on a loaded (i.e. having initial conditions) vibrating string. Both undamped pendula have cyclic trajectories i.e. an infinity of possible periodic solutions. It is shown that functional differential equations of neutral type can be associated and that the exchange of signals through the vibrating string is inducing local dampings in the pendula. The cyclic motions are replaced by a complex behavior of the overall system. This kind of behavior can have several explanations that are valid simultaneously Lepri and Pikovsky (2014): periodicity of the initial conditions of the vibrating string but also undamped modes of the nonlinear overall system. At their turn these undamped modes are a consequence of the fact that the resulting difference operator of the associated functional differential system of neutral type is not strongly stable but only critically stable – a specific aspect of almost all mechanical systems.

We shall consider here an electric analogue of the aforementioned mechanical systems: two LC oscillators having each a tunnel diode are connected to a lossless LC transmission line of infinite length. Each independent oscillator has a stable limit cycle while the LC transmission line has periodic initial conditions. Taking throughout the paper the approach of associating functional differential equations by integrating along the characteristics, what is left of it is structured as follows. First the isolated oscillator is considered, recalling the fact that it has a unique limit cycle which is orbitally stable Răsvan et al. (2008). Next there is applied integration along the characteristics for the boundary value problems that are modeling the structure with two oscillators coupled to two the LC lossless transmission line of the infinite length.

In this way a one to one correspondence is established between the solutions of the boundary value problems and the solutions of an associated system of functional differential equations of neutral type Răsvan (2014). The problem of the synchronization thus reduces to the problem of the forced oscillations for the aforementioned system of neutral functional differential equations. Here the *frequency domain method* of V.M. Popov

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will be applied, based on the basic existence result of Halanay and Răsvan (1977). The strong stability of the difference operator will turn again to be essential. In the final part of the paper some conclusions will be presented and also some continuation of the research will be suggested.

2. THE BASIC THEORY OF THE LOCAL OSCILLATORS

The essentials of this theory will be presented after Răsvan et al. (2008) and making use of the general results from Andronov et al. (1966). The electric diagram together with the nonlinear characteristic of the tunnel diode are given in Fig. 1.

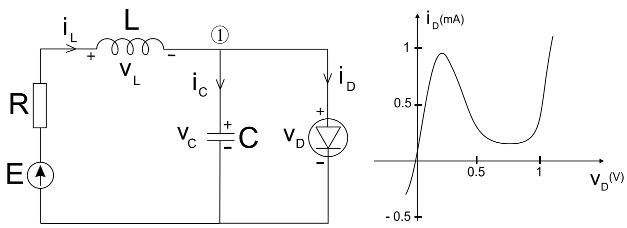

Fig. 1. Local oscillator (standard LC circuit).

It is not difficult to see that there are possible either one or three equilibria Fig. 2.

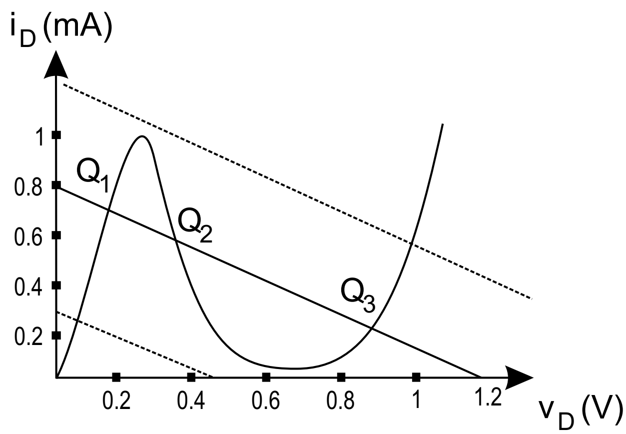

Fig. 2. Equilibria of the tunnel diode oscillator

In the case of a single equilibrium, this equilibrium can be made globally asymptotically stable i.e. the oscillator does not oscillate (just like in the morphogenetic model of A.M. Turing). Taking into account the existing results in the theory of the forced oscillations it is possible to prove existence of a periodic solution induced by the line initial conditions; this periodic solution *will display an attraction domain what will signify local synchronization*.

The case of three equilibria is even more interesting: the oscillator is unlike other ones (Liénard, van der Pol, Duffing) namely for this reason. Also the equilibrium Q_2 in the "middle" of the "load line" in Fig. 2 is a saddle point while the other two are either both focuses (most probable) or both nodes; therefore they are all inside the possible limit cycle (according to Poincaré theorem). Considering the system in deviation with respect to the saddle point, the tunnel diode characteristic becomes a S-like function for which self sustained oscillations are most likely.

Consequently we shall consider the oscillator in fig. 1 and write down its equations as follows

$$C\frac{\mathrm{d}v_C}{\mathrm{d}t} = i_C = i_L - i_D = i_L - f(i_D)\ ;\ v_C = v_D$$
$$L\frac{\mathrm{d}i_L}{\mathrm{d}t} = -Ri_L - v_C + E \tag{1}$$

where E is a d.c. voltage source for steady state bias. The equilibria (d.c. solutions) are given by the nonlinear equation

$$f(v_D) = \frac{E}{R} - \frac{v_D}{R} \tag{2}$$

and with a suitable choice of E and R this equation will have three roots: $\underline{V}$, V_*, $\bar{V}$ such that $\underline{V} < V_* < \bar{V}$ and $f'(V_*) < 1/R < 0$. Therefore the equilibrium (V_*, I_*) is always a saddle point. The other two equilibria $(\underline{V}, \underline{I})$ and $(\bar{V}, \bar{I})$ where $\bar{I} < I_* < \underline{I}$ are either stable nodes or stable foci. Therefore the necessary condition for a limit cycle (Poincaré theorem) namely $N = S + 1$ is fulfilled and the limit cycle, if it does exist, should encompass all three equilibria. Following and completing citemtns:08 we sketch the proof of the existence of the limit cycle. Introducing the deviations from the saddle point equilibrium, namely

$$\iota = i_C - I_*\ ,\ \nu = v_D - V_*$$

the equations in deviations are obtained

$$C\frac{\mathrm{d}\nu}{\mathrm{d}t} = \iota - [f(\nu + V_*) - f(V_*)] = \iota - g(\nu)$$
$$L\frac{\mathrm{d}\iota}{\mathrm{d}t} = -\nu - R\iota \tag{3}$$

Associate the Liapunov function given by the electromagnetic energy stored in the capacitor and the coil

$$\mathscr{V}(\nu, \iota) = \frac{1}{2}(C\nu^2 + L\iota^2) \tag{4}$$

whose derivative function along the solutions of (3) is

$$\mathscr{W}(\nu, \iota) = -\nu g(\nu) - R\iota^2 \tag{5}$$

We have $\underline{\nu} = \underline{V} - V_* < 0 < \bar{\nu} = \bar{V} - V_*$ and clearly $\nu g(\nu) > 0$ for $\nu \notin (\underline{\nu}, \bar{\nu})$. Therefore $\mathscr{W}(\nu, \iota) < 0$ for sufficiently large deviations. This is nothing more but *dissipativeness in the sense of Levinson* (ultimate boundedness) hence all trajectories enter a disk of radius larger than $\tilde{\nu} + \varepsilon$ and remain there, where $\tilde{\nu} = \max\{-\underline{\nu}, \bar{\nu}\}$.

On the other hand system (3) fulfils the conditions of the Theorem of Yakubovich on exponential instability Yakubovich (1970) (or Theorem 1.1.3 of Gelig et al. (1978); unfortunately this theorem was not included in the English revised version of the book Gelig et al. (2004)). Consequently any solution of (3) will leave the disk of radius $\min\{-\underline{\nu}, \bar{\nu}\} - \varepsilon$. Summarizing, all solutions, regardless the amplitude of the initial deviations, will ultimately enter an invariant annulus, which is equilibrium free. According to the theorem of Poincaré and Bendixson, the annulus contains an orbitally stable limit cycle. We have no information about the period of the corresponding self sustained oscillation but if the circuit elements are identical (or almost) for both oscillators, we may assume quite close periods. However this aspect is irrelevant for the present paper.

3. THE MATHEMATICAL MODELING OF THE COUPLED OSCILLATORS

A. We shall refer here to fig. 3 describing the circuitry of the couplings as well as that of the oscillators. The infinite length lossless LC transmission line is described by the corresponding telegraph equations while the local oscillators and their coupling to the line will define some boundary conditions. We shall have