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On the Delay Margin for Consensus in Directed Networks of Anticipatory Agents

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Abstract: We consider a linear consensus problem involving a time delay that arises from predicting the future states of agents based on their past history. In case the agents are coupled in a connected and undirected network, the exact condition for consensus is that the delay be less than a constant threshold that is independent of the network topology or size. In directed networks, however, the situation is quite different. We show that the allowable maximum delay for consensus depends on the network topology in a nontrivial way. We study this delay margin in several network constellations, including various circulant networks with directed links. We show that the delay margin depends not only on the number of neighbors, but also on the directionality of connections with those neighbors. Furthermore, the delay margin improves as the circulant networks are rewired en route to a small-world configuration.

Keywords: Consensus, prediction, anticipation, delay, network, graph theory, stability analysis.

1. INTRODUCTION

Consensus and coordination problems arise in a wide range of applications where multi-agent systems interact to agree on a common value of a certain quantity of interest. We can cite here, among others, Lynch (1996) in distributed computing, DeGroot (1974) in management science, Vicsek et al. (1995) in flocking and swarming theory, Fax and Murray (2004) in distributed control, and Olfati-Saber and Shamma (2005) in sensor networks. The classical linear consensus problem can be formulated in the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, n \tag{1}$$

where n is the number of agents in the network, $x_i \in \mathbb{R}$ is the state of the agent i at time t, which changes under the interaction with other agents, and a_{ij} are nonnegative numbers describing the interaction strength between agents i and j. Consensus can then be formally defined as follows.

Definition 1. The system (1) is said to *reach consensus* if for any set of initial conditions $\{x_i(0)\}$ there exists $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all i, in which case the number c is called the *consensus value.*

Under mild conditions related to the connectivity of the network, it can be shown that the system (1) reaches consensus from arbitrary initial conditions, and the consensus value equals the average of initial conditions of the agents. The problem becomes more interesting when the system involves a time delay τ, for example an information processing delay modeled by

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\left(x_j(t-\tau) - x_i(t-\tau)\right) \tag{2}$$

which has been studied in Olfati-Saber and Murray (2004). In this case, it is known that there exists an upper limit $\tau_{\max}$ such that the system (2) reaches consensus from arbitrary initial conditions if and only if $\tau < \tau_{\max}$ (see, for instance, Olfati-Saber and Murray (2004)). Another model, which involves an information transmission delay, is given by (Moreau, 2004; Seuret et al., 2008; Atay, 2012, 2013)

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}\left(x_j(t-\tau) - x_i(t)\right). \tag{3}$$

It has been shown that such a system reaches consensus from arbitrary initial conditions regardless of the value of the delay τ as long as the network contains a spanning tree; however, the consensus value depends on the system's history in a nontrivial way (Atay, 2012, 2013).

In this paper we are concerned with a rather different source of delay, arising from the *anticipatory nature* of the agents. More precisely, we consider a network of intelligent agents who try to predict the future states of their neighbors in their interaction. Formulating in the context of system (1), agent i uses, instead of the current state $x_j(t)$ of its neighbor, a predicted value $\hat{x}_j(t+\tau)$ of its future state, yielding

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(\hat{x}_j(t+\tau) - x_i(t)), \quad i = 1, \ldots, n \tag{4}$$

Using a *first order estimation* (Atay and Irofti, 2015), the prediction of the future can be done by a linear

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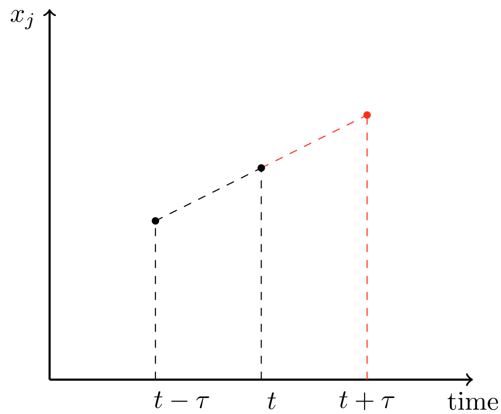

Fig. 1. Linear prediction of the future state $x_j(t+\tau)$ of an agent j using its present and past states.

extrapolation from past values, namely

$$\hat{x}_j(t+\tau) = x_j(t) + \frac{x_j(t) - x_j(t-\tau)}{\tau}\tau, \\ = 2x_j(t) - x_j(t-\tau). \quad (5)$$

(See Figure 1 for a graphical depiction.)

Using (5) in (4), we arrive at the main model that we will study in this paper:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(2x_j(t) - x_j(t-\tau) - x_i(t)), \quad (6)$$

Note that we have additionally normalized the summation term via dividing by the (generalized) degree d_i of node i, $d_i = \sum_{j=1}^{n} a_{ij}$. This normalization gives rise to a *normalized Laplace operator*, which is a natural choice in several applications and has some advantageous properties, as will be briefly reviewed in Section 2. In particular, the normalization bounds the spectrum of Laplacian regardless of the network size, thus allowing comparison of networks of very different sizes.

When the network is undirected (i.e. $a_{ij} = a_{ji}\ \forall i, j$) and connected, it has been shown (Atay and Irofti, 2015, 2016) that system (6) reaches consensus from arbitrary initial conditions if and only if

$$\tau < 1. \quad (7)$$

In other words, in the undirected case, the maximum allowable delay for consensus in (6) (the *delay margin*) equals 1 regardless of the network topology. The situation for directed networks is different, however, as we show in this paper. In particular, the network topology turns out to play an important role in affecting the delay margin.

In the following, we first prove that (7) is a necessary condition for consensus, but it is not sufficient. Moreover, as already mentioned above, the undelayed network ($\tau = 0$) always reaches consensus (as long as it contains a directed spanning tree). It follows by continuity that, sufficiently small delays will not destroy stability of the consensus. Hence, the delay margin for (6) is some positive number less than one. We calculate the locus of (complex) Laplacian eigenvalues that are detrimental for consensus. Just like undirected networks, many directed networks also turn out to enjoy (7) as the exact condition for consensus. However, we show that some specific networks that are actually frequently used in the literature do have much lower delay margins. We study these circulant networks in detail with respect to their Laplacian eigenvalues and determine their delay margins. We also consider random rewiring of circulant networks en route to small-world configuration and show that a few rewirings improve the delay margin already, although the improvement is not monotone with further rewirings.

2. DIRECTED NETWORKS AND CHARACTERISTIC ROOTS

A directed graph (or *digraph*) $G = (V, E)$ consists of a finite set V of vertices and a set of directed edges $E \subset V \times V$ consisting of ordered pairs of vertices. We consider only simple, non-trivial graphs without self-loops or multiple edges. We denote by $A = [a_{ij}]$ the (weighted) adjacency matrix of the graph, where $a_{ij} > 0$ if there is a directed link from node j to node i, and $a_{ij} = 0$ otherwise. The *in-degree* d_i of node i is defined as $d_i = \sum_{j=1}^{n} a_{ij}$, i.e., the sum of the elements of the i^{th} row of A, and $D = \text{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix of in-degrees.

Assuming that $d_i \neq 0\ \forall i$, the normalized Laplacian matrix is defined as

$$L = I_n - D^{-1}A, \quad (8)$$

where n is the number of nodes in the network and I_n is the identity matrix of size n. The normalized Laplacian L naturally arises in a class of important problems, in particular in random walks on networks, as $D^{-1}A$ is the transition matrix for probability distributions arising from such walks (Chung, 1997).

An application of Gershgorin's theorem to the definition of L shows that the Laplacian eigenvalues λ_k are complex numbers satisfying

$$|1 - \lambda_k| \leq 1, \quad k = 1, 2, \ldots, n. \quad (9)$$

Furthermore, the first eigenvalue λ_1 is always zero and corresponds to the eigenvector $(1, 1, \ldots, 1)^\top$. In the special case of undirected networks the eigenvalues are all real, because $D^{-1}A$ is similar to a symmetric matrix, $D^{-1}A = D^{-1/2}(D^{-1/2}AD^{-1/2})D^{1/2}$, as A is symmetric and D is diagonal.

In matrix form, (6) becomes

$$\dot{x}(t) = D^{-1}A(2x(t) - x(t-\tau)) - x(t), \quad (10)$$

with $x = (x_1, x_2, \ldots, x_n)^\top$. Suppose that L has a complete set of eigenvectors $\{\mathbf{v}_k\}$ corresponding to the eigenvalues $\{\lambda_k\}$. Then one can write $x(t) = \sum_{k=1}^{n} u_k(t)\mathbf{v}_k$ for some scalar coefficients u_k, which transforms (10) into a system of n decoupled scalar equations

$$\dot{u}_k(t) = (1 - 2\lambda_k)\,u_k(t) - (1 - \lambda_k)u_k(t-\tau), \quad (11)$$

for $k = 1, \ldots, n$. The characteristic equation corresponding to the eigenmode (11) is

$$\psi_k(s) := s - 2(1 - \lambda_k) + 1 + (1 - \lambda_k)e^{-s\tau} = 0, \quad (12)$$

and the characteristic equation for the whole system (10) can be written as

$$\Psi(s) := \prod_{k=1}^{n} \psi_k(s) = 0. \quad (13)$$

Note that $s = 0$ is always a characteristic root for the first factor