Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-10 (2016) 229–234

A comparative overview and expansion of frequency based stability boundary mapping methods for time delay systems

Frank Schrödel, Michael Abdelmalek, Dirk Abel *

* * *Institute of Automatic Control, RWTH Aachen University, 52074 Aachen, Germany,* {F.Schroedel, D.Abel}@irt.rwth-aachen.de

Abstract: This paper presents a comparative overview of the existing frequency based methods for the classical problem about stability boundary calculation for time delay systems. Based on this, a generalization of the problem of calculating the stability region for time delay systems is given. Moreover, a novel method for computing the stabilizing system parameters as well as the controller parameter space for time delay systems is demonstrated. For this purpose, the parameter space approach is utilized as the basic vehicle for mapping the stability bounds. This technique provides an unique possibility for the efficient calculation of parameter spaces which are not only consisting of the uncertain time delay parameter.

Keywords: Stabilizing controllers, Robust stability, Time delay, Surveys, Computational methods

1. INTRODUCTION

Time delay systems are often used to model the behaviour of systems in the field of biology, chemistry, economics, physics, population dynamic and engineering science. The first analysis of time delay systems was done by the Bernoulli brothers and L. Euler in the 18. century. With the systematic studies of A. Myshkis and R. Bellman, a deeper understanding of this systems began, see Fridman (2014). A lot of publications were produced in this field of study since 1960. The topic of robust control of time delay systems led to the time delay boom since the middle of the 1990. The last developments happened so fast that some important links between some of the individual theoretical results were lost or not recognized. All of the frequency based methods discussed in this paper depend on this continuity theorem of Datko (1978) which simply states that the linear time invariant delay system

$$\dot{x} - \sum_{m=1}^{r} B_m \dot{x}(t - m\tau) = A_0 \dot{x} + \sum_{m=1}^{r} A_m x(t - m\tau) \quad (1)$$

is stable for some values of parameters and unstable for other values. Also, there are values for these parameters in-between where the system is on the stability margin. The system is asymptotically stable if all the poles lie in the open left half plane (LHP). In the delay case, there exists an infinite number of poles which makes the determination of the stability for time delay systems more complicated. However, it can be shown that all the infinite roots behave regularly, see Hohenbichler (2003). They are located on asymptotes which are called root chains. Retarded type systems have roots which move deep into the LHP. The roots of neutral type systems lie on a vertical asymptote parallel to the imaginary axis. Forestall type systems have roots which are located deep in the right half plane (RHP) and these systems are unstabilizable. The present paper focusses on the problem of calculating the stabilizing parameter space for delay systems with uncertain system parameters. Most of the classical approaches in this field concentrate only on calculating the stabilizing delay values and not the other stabilizing system parameters. Therefore, most of these methods fail to solve this problem. Accordingly, a novel method for computing the stabilizing system parameters as well as the controller parameter space for time delay systems will be presented.

The structure of the paper is as follows. The next section deals with the problem where the time delay parameter value is the only uncertain parameter of a system. It offers a survey of recent stability calculation methods and develops a generalization of the problem of calculating the stability region for time delay systems. Thereafter, a short review of methods for the stabilizing controller parameter space calculation will be presented. Finally in section III, a new approach for the stabilizing system parameter space will be proposed by assuming the time delay as fixed.

2. DELAY PARAMETER SPACE CALCULATION

The following methods consider the controller gains as well as the system parameters as constant and the only unknown parameter is the time delay value. These methods for the delay parameter space calculation are based on the τ-decomposition concept which simply divides the delay region into intervals where each interval consists of the same number of unstable roots NU(τ). At the boundary of each interval there is at least one pole on the imaginary axis. In some cases, there is only one interval which is the whole axis. Such systems are stable/unstable independent of delay. The so called crossing frequencies $\omega_c \in \Re_+$ and crossing delays $\tau = \tau_{ck}$ are satisfying the characteristic equation $\delta(s, \tau)$ of the system for $s = j\omega_c$. The delay parameter space calculation method utilizes the paradigm cluster treatment of characteristic roots (CTCR) which

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.07.534

states that the set of all crossing frequencies $\{\omega_c\}$ consists of a finite number of crossing frequencies which can not be more than n^2 where n is the system order, see Niculescu (2004). For each crossing frequency $s = j\omega_c$ there is a cluster with an infinite number of crossing delays τ_{ck} that satisfies the characteristic equation which is related by

$$\tau_{ck} = \tau_c + 2\pi k/\omega_c \quad k = 0, 1, 2, \ldots . \tag{2}$$

The period between two crossing delays in the same cluster is $2\pi/\omega_c$. The period is smaller and the crossing rate is higher for higher crossing frequencies. The classical concept of root tendency $\mathrm{RT} = \mathrm{sign}\left[\mathrm{Re}(\partial s/\partial\tau)|_{s=j\omega_c,\tau=\tau_c})\right]$. indicates the behaviour of the poles on the imaginary axis as τ_{ck} increases from τ_{ck} to $\tau_{ck} + \epsilon$ where ϵ is a very small positive value. The invariance property states that for each cluster (ω_c, τ_{ck}) the root tendency is the same for all crossing delays, see Olgac (2002). If RT = 1, the cluster (ω_c, τ_{ck}) is destabilizing and the poles crosses from the LHP to the RHP and $NU(\tau)$ increases by 2. Other vice, the cluster is stabilizing and the poles crosses from the RHP to the LHP and $NU(\tau)$ decreases by 2. Note, a more general invariance property has been presented in Li (2015) which is applicable even for systems with multiple roots. The following subsections briefly explain the most popular delay parameter space calculation methods before stating a generalization of the these methods.

2.1 Direct method

These intuitive and straightforward computational approach was introduced in Walton (1987); Cooke (1986). A necessary and sufficient condition for determining at which values of delay the roots lie on the imaginary axis is that $s = j\omega$ is the solution of the characteristic equation

$$\delta(j\omega, \tau) = a_0(j\omega) + a_1(j\omega)e^{-j\omega\tau} = 0 \tag{3}$$

where $\omega \in \Re_+$. Using the fact that the roots always cross the imaginary axis as a complex conjugated pair, the substitution $s = -j\omega$ also satisfies the equation

$$\delta(-j\omega, \tau) = a_0(-j\omega) + a_1(-j\omega)e^{j\omega\tau} = 0. \tag{4}$$

Multiplying both equations yield

$$W(\omega) = a_0(j\omega)a_0(-j\omega) - a_1(j\omega)a_1(-j\omega) = 0 \tag{5}$$

which is a polynomial in ω with finite dimension of degree n^2. A similar decoupling idea is also used in Bhattacharyya (2012) and can be traced back to Tsypkin (1946). Only positive solutions of ω form the crossing frequency set $\{\omega_c\}$. The solution of $W(\omega)$ does only depend on the system parameters and not on the delay. If ω_c is sorted in descending order, it holds for the single delay case that the first crossing frequency is destabilizing, the second stabilizing and so on, see Walton (1987). This property can be used instead of calculating RT. The solution for the associated τ_{ck} values with each ω_c can be obtained using real and imaginary parts of equation (3)

$$\cos(\omega_c\tau) = \mathrm{Re}(-a(j\omega_c)/b(j\omega_c)) \tag{6}$$

$$\sin(\omega_c\tau) = \mathrm{Im}(a(j\omega_c)/b(j\omega_c)). \tag{7}$$

Solving for τ yields

$$\tau_{ck} = \tan^{-1}\left(\frac{\mathrm{Im}\{\frac{a(j\omega_c)}{b(j\omega_c)}\}}{\mathrm{Re}\{-\frac{a(j\omega_c)}{b(j\omega_c)}\}}\right) + \frac{2\pi k}{\omega_c},\ k = 0, , 1, 2, \ldots .$$

There are no crossings as τ increases and the system is stable/unstable independent of delay if there are no real solutions for $W(\omega)$ existing. The system is unstable for all values of τ if $s = 0$ is a solution for the delay free system $a(s) + b(s) = 0$. This method can also be extended to multiple commensurate delays (see Walton (1987)) which could produce additional fictitious crossing frequencies (see Sipahi (2006)).

2.2 Rekasius substitution method

This method was firstly introduced in Rekasius (1980). By applying the simple substitution

$$e^{-\tau s} = \frac{1 - Ts}{1 + Ts},\quad T \in \Re, \tau \in \Re_+ \tag{8}$$

the quasi-polynomial of the infinite dimension is transformed into a finite dimensional polynomial of order n^2:

$$\delta(s, T) = \sum_{k=0}^{n} a_k(s)(1 + Ts)^{n-k}(1 - Ts)^k = 0. \tag{9}$$

This substitution works only at crossing frequencies when $s = j\omega$. The new finite dimension characteristic equation has poles on the imaginary axis that coincide with the infinite one. A Routh-Hurwitz table can be formed by using equation (9). To get values of T_c that cause poles on the imaginary axis, it is required to solve the term $R_{11}(T)$ in front of s^1 for real values of T and to substitute these values into the two terms $R_{21}(T)$ and $R_{22}(T)$ in front of s^2, see Niculescu (2004). If the two terms have the same sign then these values of T should be included in $\{T_c\}$. The crossing frequencies are given by $\omega_c = \sqrt{R_{22}/R_{21}}$ which is considered as an one to one mapping from T_c to ω_c. The mapping of T_c and ω_c to τ_{ck} which can be derived easily by equating the phases in equation (8) is

$$\tau_{ck} = 2/\omega_c(\tan^{-1}(\omega_c T_c) + k\pi) \quad k = 0, 1, 2, \ldots . \tag{10}$$

It is extended to systems with two time delays in Sipahi (2004, 2005) .

2.3 Matrix multiplication method

The Matrix multiplication method was introduced in Lousiell (2001). It eliminates the delay and solves the resulting equation for the crossing frequencies for time delay systems described in matrix form. The characteristic equation for $r = 1$ (see equation (1)) is

$$\delta(s, \tau) = \det(sI - A_0 + (-sB_1 - A_1)e^{-\tau s}) = 0. \tag{11}$$

This can be seen as the following eigenvalue problem

$$(sI - A_0)v = e^{-\tau s}(sB_1 + A_1)v, \qquad v \neq 0 \tag{12}$$

conjugating and transposing the previous equation gives

$$v^*(sI + A_0^T) = e^{\tau s}v^*(sB_1^T - A_1^T). \tag{13}$$

Multiplying both equations eliminates the delay and the resulting equation is

$$(sI - A_0)vv^*(sI + A_0^T) = (sB_1 + A_1)vv^*(sB_1^T - A_1^T). \tag{14}$$

The previous equation can be written as

$$((sI - A_0)\otimes(sI + A_0) - (sB_1 + A_1)\otimes(sB_1 - A_1))u = 0 \tag{15}$$

where $u = \xi vv^*$, using the mapping ξ in Lousiell (2001). Equation (15) can be seen as an eigenvalue problem of

$$\det(sE - J) = 0 \qquad \text{with}$$
$$E = \begin{bmatrix} I \otimes I & B_1 \otimes I \\ I \otimes B_1 & I \otimes I \end{bmatrix}, J = \begin{bmatrix} A_0 \otimes I & -A_1 \otimes I \\ I \otimes A_1 & -I \otimes A_0 \end{bmatrix}. \tag{16}$$

Now, the crossing frequency set $\{\omega_c\}$ of equation (11) can be calculated by computing the purely imaginary