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LQ Control of Lotka-Volterra Systems Based on their Locally Linearized Dynamics ⋆,⋆⋆

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Abstract: This work applies the LQ control framework to the class of quasi-polynomial and Lotka-Volterra systems through the linearized version of their nonlinear system model. The primary aim is to globally stabilize the original system with a suboptimal LQ state feedback by means of a well-known entropy-like Lyapunov function that is related to the diagonal stability of linear systems. This aim can only be reached in the case when the quasi-monomial composition matrix is invertible. In the rank-deficient case only the local stabilization of the system is possible with an LQ controller that is designed using the locally linearized model of the closed-loop system model.

Keywords: Lyapunov stability, nonlinear systems, LQ control, linear matrix inequalities

1. INTRODUCTION

A wide range of nonlinear systems can only be tackled using nonlinear techniques (Isidori (1995)). The majority of such techniques are applicable only for a narrow class of nonlinear systems, while the more generally applicable methods suffer from computational complexity problems. One possible way of balancing between general applicability and computational feasibility is to find nonlinear system classes with good descriptive power but well characterized structure, and utilize this structure when developing control design methods. This is possible, for example, in the case of quasi-polynomial systems, that is the subject of this paper.

Previous work in the field of quasi-polynomial systems include the paper of Figueiredo et al. (2000), which gives a sufficient condition for the global stability of quasi-polynomial systems in terms of the feasibility of a linear matrix inequality (LMI). Based on this result, it has been shown in Magyar et al. (2008), that the globally stabilizing state feedback design for quasi-polynomial systems is equivalent to a bilinear matrix inequality. It is also shown there, that although the solution of a bilinear matrix inequality is an NP hard problem, an iterative LMI algorithm could be used. A summary of linear and bilinear matrix inequalities and the available software tools for solving them can be found in VanAntwerp and Braatz (2000).

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Another control synthesis algorithm for polynomial systems is presented in Tong et al. (2007). A different approach has been presented in Magyar and Hangos (2015) where Lotka-Volterra models has been globally stabilized based on their underlying linear model.

The aim of this paper is to apply a *LQ based state feedback controller* for quasi-polynomial and Lotka-Volterra systems through a locally linearized model corresponding to a (unique) positive equilibrium point of the closed-loop system. The primary aim is the formulation of a LQ problem that yields a diagonally stable LTI system and the corresponding globally asymptotically stable Lotka-Volterra or quasi-polynomial system. Of course, the case when the quasi-monomial composition matrix is rank-deficient, it is far from being trivial and one can expect only local asymptotic stability in this case.

2. BASIC NOTIONS

The most important results on quasi-polynomial (QP) and Lotka-Volterra (LV) systems and on their stability analysis are briefly presented here.

2.1 Quasi-Polynomial and Lotka-Volterra Systems

The system dynamics of an *autonomous quasi-polynomial* (QP) system can be described by a set of differential-algebraic equations (DAEs), where the ordinary differential equations

$$\frac{dz_i}{dt} = z_i \left(\lambda_i + \sum_{j=1}^{m} \alpha_{ij} q_j \right) \ , \ i = 1, ..., n, \tag{1}$$

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are equipped by the so called quasi-monomial relationships

$$q_j = \prod_{i=1}^{n} z_i^{\beta_{ji}}, \qquad (2)$$

that are apparently nonlinear (monomial-type) algebraic equations. Two sets of variables are defined, that are (i) the differential variables z_i, $i = 1, ..., n$, and (ii) the quasi-monomials q_j, $j = 1, ..., m$. The parameters of the above model are collected in the coefficient matrix $[A]_{ij} = \alpha_{ij}$, *quasi-monomial composition matrix* $[B]_{ji} = \beta_{ji}$ and a vector $[\lambda]_i = \lambda_i$. Then equation (1) can be written in the compact form

$$\dot{z} = D(z)\,(\lambda + A\,q)\,, \qquad (3)$$

where $D(\cdot)$ stands for $diag(\cdot)$.

It is easy to see that Lotka-Volterra systems form a special subset of the quasi-polynomial systems with the choice $B = I$, and thus $q = z$ with $n = m$

$$\dot{z} = D(z)\,(\lambda + A\,z)\,. \qquad (4)$$

This constitutes a special square invertible case for the quasi-monomial composition matrix B.

Lotka-Volterra form It can be shown (see Hernández-Bermejo and Fairén (1995)) that the class of QP systems is closed under the so called quasi-monomial transformation (QM transformation), where the product $M = BA$ remains constant when transforming a QP model. This way the QM transformation splits the set of QM models into equivalence classes that are represented by a Lotka-Volterra model where the differential variables are the quasi-monomials

$$\dot{q} = D(q)(B\,\lambda + BAq) = D(q)(B\,\lambda + Mq), \qquad (5)$$

where q satisfy the algebraic equations (2).

We can consider the logarithm of these algebraic equations because of the positivity of the two sides

$$\underline{\ln}\ q = B \cdot \underline{\ln}\ z, \qquad (6)$$

where $[\underline{\ln}\ x]_i = \ln\ x_i$. Then (6) is equivalent to

$$\underline{\ln}\ q \in \text{range}(B). \qquad (7)$$

This manifold (7) is an invariant subspace of the dynamics (5) because

$$\frac{d\underline{\ln}\ q}{dt} = B\,(\lambda + Aq) \in \text{range}(B). \qquad (8)$$

It is easy to see that when the matrix B is invertible then

$$z = \exp(B^{-1}\,\underline{\ln}\ q) \qquad (9)$$

for all $q \in \mathbb{R}^n_{>0}$. It means that the algebraic equation (2) has a positive solution for all $q \in \mathbb{R}^n_{>0}$.

In the usual case of $m > n$, the right side of the transformed ODE (5) would be simpler, but we have to consider the algebraic conditions (2).

Steady-state points The non-zero steady-state point(s) of the dynamic equations (1) are obtained by setting the left-hand sides equal to zero, and solve the equations

$$\mathbf{0} = \lambda + A \cdot q^*, \qquad (10)$$

for q^* (the vector q^* has a quasi-monomial relationship with the equilibrium point z^*). *Generally, this equation has a unique solution if* A is quadratic and invertible, but the solution is not necessarily positive.

Otherwise, if $m > n$, then the set of equations (10) may have infinitely many solutions. However, the set of algebraic equations (6) puts a set of nonlinear constraints to the elements of the vector q^* (i.e. the vector q should be taken from a lower dimensional manifold of the quasi-monomial space) that may result in a unique equilibrium point even in this case. The existence of strictly positive solutions without algebraic constraints can be tested by various algorithms or simple linear programming.

Quasi-Polynomial models with input Let us consider a linear input structure for the original QP model (1), that can be formally derived by regarding λ as a function of the input vector u

$$\lambda = \phi\,u, \quad u \in \mathbb{R}^p, \quad \phi \in \mathbb{R}^{n\times p}, \quad p \le n \qquad (11)$$

such that the state equation is in the form

$$\dot{z} = D(z)\,(\phi\,u + A\,q) \qquad (12)$$

with the algebraic equations (6).

2.2 Stability Condition of QP systems

Assume that there exists a positive steady-state point z^* of the QP system (1). Then this steady state point is globally asymptotically stable if there exists a positive diagonal matrix P for the product matrix $M = BA$ such that

$$MP + PM^T < 0, \qquad (13)$$

or

$$QM + M^TQ < 0, \qquad (14)$$

where $Q = P^{-1}$ is positive definite diagonal matrix (Gléria et al. (2001); Figueiredo et al. (2000)). In this case, the matrix M is called diagonally stable (Kaszkurewicz and Bhaya (2012)).

It is important to note that the feasibility of the above LMI is a sufficient (but not always necessary) condition for the stability, as it is derived from the dissipativity property of the entropy-like Lyapunov function

$$V(q) = \sum_{i=1}^{m} \gamma_i \left(q_i - q_i^* - q_i^* \ln \frac{q_i}{q_i^*} \right)$$

where $\gamma_i > 0$ and $Q = D(\gamma)$.

2.3 Locally Linearized QP Model

The linearized version of the QP model (1) around its positive equilibrium point z^* is in the form

$$\frac{\Delta z}{dt} = \left[D(z^*)\,A\,D(q^*)\,B\,D(z^*)^{-1} \right] \Delta z \qquad (15)$$

where $\Delta z = z - z^*$. When the matrix B is invertible, then we can transform (15) with the linear transformation

$$x = \left[D(q^*)\,B\,D(z^*)^{-1} \right] \Delta z = T\Delta z \qquad (16)$$

with the transformation matrix

$$T = D(q^*)\,B\,D(z^*)^{-1}\,, \qquad (17)$$

and the transformed system is in the form

$$\dot{x} = D(q^*)\,B\,A\,x = M^*x. \qquad (18)$$