# Optimal Linear Data Fusion for Systems with Missing Measurements

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**Abstract:**
In this paper, we provide the optimal data fusion filter for linear systems suffering from possible missing measurements. The noise covariance in the observation process is allowed to be singular which requires the use of generalized inverse. The data fusion process is made on the raw data provided by two sensors observing the same entity. Each of the sensors is losing the measurements in its own data loss rate. The data fusion filter is provided in a recursive form for ease of implementation in real-world applications.

*Keywords:* Data fusion, Kalman filter, generalised inverse.

## 1. INTRODUCTION

Multi-sensor data fusion is defined as the process of combining data from multiple resources to improve the quality of the estimation process. The data resources may be chosen to be of the same type or of different types. The first case increases system reliability and the second case increases the knowledge about the system. The multi-sensor data fusion has been investigated by many research bodies, Hall (1992),Hall and Llinas (1997) for a typical literature review. Most of the research about multi-sensor data fusion considers the development of algorithms and architectures that increase the usability and feasibility of the estimation process, however, they assumed perfect sensor conditions and perfect communication channels. In this research we handle the problem of data fusion within the non-ideal case of uncertain observations. The sensors involved in the data fusion process are subject to random losses of measurements. The loss in measurements can be due to individual sensor conditions or interruptions in the communication channels.

The problem of missing measurements has been investigated by many researchers. It was first addressed for a class of linear filters by Nahi in Nahi. (1969) who obtained the optimal state estimator for systems with missing measurements. In the former reference, the uncertainty in the observations was assumed to be independent and identically distributed (i.i.d.). The work in Jaffer and Gupta. (1971) has generalized the work of Nahi where the uncertainty is not necessarily i.i.d. The work in Carazo and Perez (1994) extended Nahi's work to the case where the noise from the state and the noise from the measurements are correlated. Later, the missing measurements problem was considered from covariance assignment viewpoint in NaNacara and Yaz (1994). In the latter reference, all the possible estimation error covariance were characterized and an upper bound on these estimation error covariances was presented. In Nilsson and Bernhardsson (1998) and Costa and Guerra (2002), the problem was formulated as a jump linear system (JLS) switching between an open-loop configuration and a closed-loop configuration. The proposed new formulation enabled the authors in Nilsson and Bernhardsson (1998) and Costa and Guerra (2002) to study the convergence criteria of the expected estimation error covariance. However, they restricted the Kalman gain to be constant.

In recent years, the work in Yang et al. (2002) proposed the formulation of the problem as a solution to two discrete Ricatti difference equations. Using this formulation, they found the sufficient conditions for the filter that guarantees an optimized upper-bound on the estimation error covariance. The proper formulation of the convergence of the state error covariance was provided in Sinopoli et al. (2004). The problem of missing measurements was considered for a class of discrete-time linear systems that suffer from modeling uncertainties in Wang et al. (2005).

The main difference between the aforementioned research and the proposed results is the consideration of the missing measurements problem in the context of multi-sensor data fusion. This viewpoint of the problem required new formulations and new derivations for the estimation filters.

The rest of the paper is organised as follows. The next section presents the results of deriving the data fusion filter for measurements from two linear sensors suffering from missing measurements. Simulation results will be provided in section 3. Section 4 concludes the paper.

## 2. OPTIMAL DATA FUSION FILTER

In this section, the optimal data fusion of data obtained by two sensors will be presented. Generally, there are two different situations:

Situation 1:

Both sensors have the same measurements arrival rate $\beta_k$ and sequence of missing measurements.

Situation 2:

Each sensor has its own measurements arrival rate $\beta_k^{(1)}$,$\beta_k^{(2)}$. In this paper, we will consider this situation.

*2.1 Generalized Inverse*

Since the proposed technique does not require the noise covariance of the observation process to be non-singular, the use of the generalized inverse approach will be needed. The generalized inverse is defined as follows:

*Definition 1.* A generalized inverse of a matrix $A$ is a matrix $A^{\#}$ such that

$$AA^{\#}A = A$$

and if $A$ is non-singular then

$$A^{\#} = A^{-1}$$

The following lemma is important for applying the generalized inverse technique

*Lemma 2.* (Elliot and van der Hoek. (April 2006)). The generalized inverse of the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the matrix $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ where

$$\begin{aligned} E &= A^{\#} + A^{\#}BS_A^{\#}CA^{\#} \\ F &= -A^{\#}BS_A^{\#} \\ G &= -S_A^{\#}CA^{\#} \\ H &= S_A^{\#} \end{aligned}$$

and $S_A = D - CA^{\#}B$.

*2.2 Solution to Situation 2: Each Sensor Has Different Data Loss Rate*

In this section, we will provide the solution to the problem of data fusion for measurements obtained from linear filters subject to missing measurements. It is assumed that each sensor has a different data loss rate.

*Problem Formulation and Assumptions* The system under consideration is defined as

$$x_{k+1} = A_{k+1}x_k + w_{k+1} \tag{1}$$

and the observation process definition is

$$y_k^{(12)} = \gamma_k^{(12)}H_k^{(12)}x_k + v_k^{(12)} \tag{2}$$

where $\gamma_{k+1}^{(12)} = \begin{bmatrix} \gamma_{k+1}^{(1)} & 0 \\ 0 & \gamma_{k+1}^{(2)} \end{bmatrix}$

and let $\beta_{k+1}^{(12)} = \begin{bmatrix} \beta_{k+1}^{(1)} & 0 \\ 0 & \beta_{k+1}^{(2)} \end{bmatrix}$.

where $y_k^{(i)}$,$i = 1, 2$ is the output from the first and the second sensor, $w_{k+1}$ and $v_k^{(i)}$ are the uncorrelated noise sequences with zero mean and covariance $\mathbf{E}[w_k w_l^T] = Q_k\delta_{kl}$ and $\mathbf{E}[v_k^{(i)}v_l^{(i)T}] = R_k^{(i)}\delta_{kl}$ where $\delta_{kl}$ is the Kronecker delta function and $H_k$ is the sensor processing matrix. It is assumed that

$$\begin{aligned} \mathbf{E}[x_0 w_k] &= 0 \\ \mathbf{E}[x_0 v_k^{(i)}] &= 0 \\ \mathbf{E}[v_k^{(i)}v_k^{(j)}] &= 0 \\ \mathbf{E}[v_k^{(i)}w_l] &= 0 \\ \hat{x}_0 &= x_0 \\ \mathbf{E}[x_0 x_0^T] &= P_0 \end{aligned}$$

These assumptions assigns the initial values for the state estimate and error covariance. The assumptions also state that there is no correlation between the state and the noises or between the noise from the system and the noise from the sensor.

The variable $\gamma_{k+1}^{(i)}$ is a Bernoulli sequence taking values 0 and 1 randomly, $i = 1, 2$ referencing the sensor number and $\mathbf{E}[\gamma_k^{(i)} = 1] = \beta_k^{(i)}$, where $\beta_k^{(i)}$ is the percentage of measurements that contain the signal and the noise and it is assumed known in prior as it can be estimated approximately by simulation sessions.

*Unbiased Data Fusion Filter Form*

*Lemma 3.* The filter form that guarantees estimation unbiasedness will be

$$\begin{aligned} \hat{x}(k+1|k+1)^{(12)} = A_{k+1}\hat{x}(k|k)^{(12)} + K_{k+1}^{(12)}[y_{k+1}^{(12)} \\ -\beta_{k+1}^{(12)}H_{k+1}^{(12)}\hat{x}(k+1|k)^{(12)}] \end{aligned} \tag{3}$$

*Computation of the Optimal Filter Gain $K_{k+1}^{(12)}$*

*Lemma 4.* The optimal filter gain $K_{k+1}^{(12)}$ will be of the form

$$K_{k+1}^{(12)} = C_{k+1}^{(12)}D_{k+1}^{(12)\#} \tag{4}$$

where $C_{k+1}^{(12)} = [\, L_1 \;\; L_2 \,]$

$$\begin{aligned} L_1 = \mathbf{E}[&(x_{k+1} - A_{k+1}\hat{x}(k|k)^{(12)})(y_{k+1}^{(1)} \\ &-\beta_{k+1}^{(1)}H_{k+1}^{(1)}\hat{x}(k+1|k)^{(12)})^T] \\ =\, &\beta_{k+1}^{(1)}A_{k+1}P_k^{(12)}A_{k+1}^T H_{k+1}^{(1)T} + \beta_{k+1}^{(1)}Q_{k+1}H_{k+1}^{(1)} \end{aligned} \tag{5}$$

$$\begin{aligned} L_2 = \mathbf{E}[&(x_{k+1} - A_{k+1}\hat{x}(k|k)^{(12)})(y_{k+1}^{(2)} \\ &-\beta_{k+1}^{(2)}H_{k+1}^{(2)}\hat{x}(k+1|k)^{(12)})^T] \\ =\, &\beta_{k+1}^{(2)}A_{k+1}P_k^{(12)}A_{k+1}^T H_{k+1}^{(2)T} + \beta_{k+1}^{(2)}Q_{k+1}H_{k+1}^{(2)} \end{aligned} \tag{6}$$

Finding value of $D_{k+1}^{(12)}$, let $D_{k+1}^{(12)} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$

$$\begin{aligned} L_{11} = \mathbf{E}[&(y_{k+1}^{(1)} - \beta_{k+1}^{(1)}H_{k+1}^{(1)}\hat{x}(k|k)^{(12)})(y_{k+1}^{(1)} \\ &-\beta_{k+1}^{(1)}H_{k+1}^{(1)}\hat{x}(k|k)^{(12)})^T] \\ =\, &\beta_{k+1}^{(1)2}H_{k+1}^{(1)}A_{k+1}P_k^{(12)}A_{k+1}^T H_{k+1}^{(1)} + \beta_{k+1}^{(1)}(1 \end{aligned}$$