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IFAC-PapersOnLine 49-9 (2016) 056–061

A Structure-preserving Model Reduction Algorithm for Dynamical Systems with Nonlinear Frequency Dependence

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Abstract: Very large-scale dynamical systems, even linear time-invariant systems, can present significant computational difficulties when used in numerical simulation. Model reduction is one response to this challenge but standard methods often are restricted to systems that are presented as standard first-order realizations; in the frequency domain such systems will be linear in the frequency parameter. We consider here dynamical systems with a *nonlinear* frequency dependence; systems for which either a standard first-order realization is unknown or inconvenient to obtain. Such systems may nonetheless have realizations that reflect important structural features of the model and we may wish to retain this structure in any reduced model used as a surrogate. In this work, we present a structure-preserving model reduction algorithm for systems having quite general nonlinear frequency dependence. We take advantage of recent algorithms that produce high quality rational interpolants to transfer functions that only require transfer function evaluation, thus allowing for nonstandard realizations that are nonlinear in the frequency parameter. However, our final reduced model will have a structure that reflects the structure of the original system, and indeed, may not have a rational transfer function. We illustrate our approach on a benchmark problem that offers a transcendental transfer function.

Keywords: dynamical systems, model reduction, structure-preservation, interpolation, $\mathcal{H}_2$ approximation, nonlinear frequency dependency, generalized realization, Loewner framework

1. INTRODUCTION

Direct numerical simulation of dynamical systems plays a crucial role (and at times may be the only option) in studying a great variety of complex physical phenomena with applications ranging from signal propagation in the nervous system, to heat dissipation in complex microelectronic devices, to vibration suppression in large wind turbines, and to timely prediction of storm surges before an advancing hurricane. However, the ever-present need for greater model resolution leads to the inclusion of ever-greater detail at the modeling stage. The resulting large-scale dynamical systems, even those that are 'simple' linear time-invariant systems, can present significant computational difficulties when used in numerical simulation due to their sheer size; hence there is a persistent need to approximate such large complex dynamical systems with smaller, yet high accuracy, approximations. This is the goal of model reduction: one constructs simpler (reduced order) models, which are much easier and faster to simulate (hence requiring far fewer computational resources) while retaining characteristics close to the original system. These simpler reduced models can then serve as efficient surrogates for the original, replacing them as components in larger systems; facilitating rapid development of controllers for real time applications; and enabling optimal system design and uncertainty analysis.

In this paper, we will focus on model reduction of stable single-input/single-output (SISO) linear dynamical systems whose behavior is described via a transfer function $\mathcal{H}(s)$. The extension of our approach to multi-input/multi-output systems may be developed in an entirely analogous and straightforward way, but here we will only focus on the SISO case to keep the presentation both simple and concise. We assume that the transfer function $\mathcal{H}(s)$ of the underlying system has a generalized realization

$$\mathcal{H}(s) = \mathcal{C}(s)^T \mathcal{K}(s)^{-1} \mathcal{B}(s) \tag{1}$$

where $\mathcal{C} : \mathbb{C} \to \mathbb{C}^n$, $\mathcal{B} : \mathbb{C} \to \mathbb{C}^n$, $\mathcal{K} : \mathbb{C} \to \mathbb{C}^{n\times n}$ are analytic in the right half plane, and $\mathcal{K}(s)$ has full rank throughout the right half plane.

The state space dimension of the underlying dynamical system typically is n and we will refer to this as the *order* of the system, even though $\mathcal{H}(s)$ may have a significantly larger (even infinite) number of poles. We are interested in settings where n could reach hundreds of thousands or more. We note that the state-space representation in (1) allows a much richer set of dynamical systems than those represented by standard first-order realizations, i.e., $\mathcal{H}(s) = \mathbf{C}(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$ where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, and $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$ are all constant quantities. One prominent example of the form in (1) are systems with internal delays such as $\mathcal{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_0 - \mathbf{A}_1 e^{-\tau s})^{-1}\mathbf{B}$ where τ is the internal delay. Note that this transfer function cannot be represented in a standard first-order form with finite dimensional choices for $\mathbf{E}$, $\mathbf{A}_0$, and $\mathbf{A}_1$. In §4, we consider two models in the form of (1). For example, the model in §4.1 has a transfer function of the form $\mathcal{H}(s) = \mathbf{C}((e^s -$

10.1016/j.ifacol.2016.07.492

$1)\mathbf{A}_2 + s^2\mathbf{A}_1 + \mathbf{I})^{-1}\mathbf{B}$ where $\mathbf{A}_2$ and $\mathbf{A}_1$ are constant matrices, and $\mathbf{C}$ and $\mathbf{B}$ are constant vectors.

Our goal will be to generate, for some $r \ll n$, a reduced dynamical system using a Petrov-Galerkin projection: select $\mathcal{V}_r \in \mathbb{R}^{n\times r}$ and $\mathcal{W}_r \in \mathbb{R}^{n\times r}$ such that $\mathcal{W}_r^T\mathcal{K}(s)\mathcal{V}_r$ is nonsingular in the right-half plane (ideally). Then, the reduced transfer function is

$$\mathcal{H}_r(s) = \mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s) \tag{2}$$

where $\mathcal{C}_r(s) = \mathcal{C}(s)\mathcal{V}_r \in \mathbb{C}^r$; $\mathcal{B}_r(s) = \mathcal{W}_r^T\mathcal{B}(s) \in \mathbb{C}^r$; and $\mathcal{K}_r(s) = \mathcal{W}_r^T\mathcal{K}(s)\mathcal{V}_r \in \mathbb{C}^{r\times r}$. Consider the aforementioned delay example $\mathcal{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_0 - \mathbf{A}_1e^{-\tau s})^{-1}\mathbf{B}$. In this case, the reduced transfer function will have the form $\mathcal{H}_r(s) = \mathbf{C}\mathcal{V}_r(\mathcal{W}_r^T\mathbf{E}\mathcal{V}_r - \mathcal{W}_r^T\mathbf{A}_0\mathcal{V}_r - \mathcal{W}_r^T\mathbf{A}_1\mathcal{V}_re^{-\tau s})\mathcal{W}_r^T\mathbf{B}$. The associated reduced dynamical model preserves the original system structure yet the dynamics evolve in a much smaller state-space. We will choose $\mathcal{W}_r$ and $\mathcal{V}_r$ so as to enforce rational interpolation conditions; $\mathcal{H}_r(s)$ will interpolate $\mathcal{H}(s)$ at selected points in the complex plane.

We will assume that $\mathcal{H}(s)$ is an $\mathcal{H}_2$-function where $\mathcal{H}_2$ denotes the set of complex functions $\mathcal{H}(s)$ that are analytic in the open right half plane $\{s = x + y \in \mathbb{C} \ : \ x > 0\}$ and such that $\sup_{x>0}\int_{-\infty}^{+\infty} |\mathcal{H}(x + \imath y)|^2 dy < \infty$. Note that $\mathcal{H}_2$ is a Hilbert space with the inner product

$$\langle \mathcal{G}, \mathcal{H}\rangle_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{H}^*(\imath\omega)\mathcal{G}(\imath\omega)d\omega \tag{3}$$

and the corresponding norm

$$\|\mathcal{H}\|_{\mathcal{H}_2} = \sqrt{\langle \mathcal{H}, \mathcal{H}\rangle_{\mathcal{H}_2}} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\ \mathcal{H}(\imath\omega)\ |^2\ d\omega}. \tag{4}$$

We will look for structure-preserving reduced order models that are high-fidelity approximations in the $\mathcal{H}_2$ norm.

The rest of the paper is organized as follows: In §2, we give the necessary background for model reduction that allows passage from generalized realizations as in (1) to structure-preserving reduced models as in (2). In §3, we introduce an algorithm for determining structure-preserving interpolatory reduced models that are high-fidelity approximations with respect to the $\mathcal{H}_2$ norm. We present two numerical examples in §4.

2. INTERPOLATORY MODEL REDUCTION OF GENERALIZED REALIZATIONS

We review here the basic interpolatory framework for building reduced models for generalized realizations having the form in (1) that preserve structure as in (2). These tools form the foundation of our approach described in §3.

2.1 Interpolatory Projections for Generalized Realizations

For linear dynamical systems with generic first-order realizations, i.e., $\mathcal{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, various model reduction methods exist to produced high-fidelity or in some cases optimal reduced models. Examples include Gramian based methods such as Balanced Truncation (Mullis et al. (1976); Moore (1981)) and Hankel Norm Approximation (Glover (1984)), or interpolatory methods such as Iterative Rational Krylov Algorithm (Gugercin et al. (2008)). For more details on model reduction of standard first-order systems, we refer the reader to Antoulas (2005); Beattie and Gugercin (2015); Baur et al. (2014); Antoulas et al. (2010) and the references there in. However, for generalized realization $\mathcal{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$ with nonlinear frequency dependence throughout the state-space quantities, many of the generic approaches do not apply except for special cases such as for second-order structure $\mathcal{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{K} + \mathbf{G})^{-1}\mathbf{B}$; see, e.g., Bai and Su (2005); Su and Craig Jr (1991); Chahlaoui et al. (2005); Meyer and Srinivasan (1996); Reis and Stykel (2008). See also recent work of Breiten (2015) using frequency domain formulations of Gramians for integro-differential equations.

Here we use interpolation. Beattie and Gugercin (2009) established the framework for interpolatory model reduction of systems with a generalized realization. The following result describes how to construct reduced-order interpolants via projection.

Theorem 1. (Beattie and Gugercin, 2009, Theorem 3) Let $\mathcal{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$ be given as in (1). Suppose r distinct points, $\{s_i\}_{i=1}^r$, are chosen in the right half plane. Define $\mathcal{V}_r \in \mathbb{C}^{n\times r}$ and $\mathcal{W}_r \in \mathbb{C}^{n\times r}$ as:

$$\mathcal{V}_r = [\mathcal{K}(s_1)^{-1}\mathcal{B}(s_1), \cdots, \mathcal{K}(s_r)^{-1}\mathcal{B}(s_r)]$$

and

$$\mathcal{W}_r^T = \begin{bmatrix} \mathcal{K}(s_1)^{-1}\mathcal{C}(s_1)^T \\ \vdots \\ \mathcal{K}(s_r)^{-1}\mathcal{C}(s_r)^T \end{bmatrix}.$$

Define $\mathcal{K}_r(s) = \mathcal{W}_r^T\mathcal{K}(s)\mathcal{V}_r$ and assume that $\mathcal{K}_r(s_i)$ is nonsingular for $i = 1, 2, 3, ..., r$. Define further:

$$\mathcal{B}_r(s) = \mathcal{W}_r^T\mathcal{B}(s), \ \text{ and } \ \mathcal{C}_r(s) = \mathcal{C}(s)\mathcal{V}_r. \tag{5}$$

Then with $\mathcal{H}_r(s) = \mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s)$ we have

$$\mathcal{H}(s_i) = \mathcal{H}_r(s_i) \ \text{ and } \ \mathcal{H}'(s_i) = \mathcal{H}_r'(s_i) \tag{6}$$

for $i = 1, ..., r$ where $\mathcal{H}'(s_i)$ denotes the first derivative of $\mathcal{H}(s)$ evaluated at s_i and $\mathcal{H}_r'(s_i)$ denotes the first derivative of $\mathcal{H}_r(s)$ evaluated at s_i.

Once the interpolation points have been selected, Theorem 1 describes precisely how one may construct a *structure-preserving interpolant* via a Petrov-Galerkin projection.

2.2 Loewner framework for Hermite Interpolation

The projection-based framework of Theorem 1 requires access to internal dynamics to construct the model reduction bases $\mathcal{V}_r$ and $\mathcal{W}_r$. However in some applications, one does not have access to a description of internal dynamics and only transfer function measurements are available. The Loewner framework of Mayo and Antoulas (2007) resolves this issue: It only requires evaluation of the transfer function (and its derivative $\mathcal{H}'(s)$ if interpolation points are repeated). As long as the transfer function $\mathcal{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$ can be sampled, one can produce a *rational* reduced model that interpolates the original system. Given a transfer function $\mathcal{H}(s)$ and a set of initial points $\{s_i\}_{i=1}^r$, define

$$(\mathbf{E}_r)_{i,j} := \begin{cases} -\dfrac{(\mathcal{H}(s_i) - \mathcal{H}(s_j))}{s_i - s_j} & \text{if } i \neq j \\ -\mathcal{H}'(s_i) & \text{if } i = j \end{cases} \tag{7}$$