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# System Degeneracy and the Output Feedback Problem: Parametrisation of the Family of Degenerate Compensators

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**Abstract:** The paper provides a new characterisation of constant and dynamic degenerate compensators for proper multivariable systems. The motivation stems from the very important property that degenerate feedback gains may be used for the linearisation of the pole assignment map and enable frequency assignment. The objective is the characterisation and parametrisation of all feedback gains that may allow the asymptotic linearisation of the pole placement map. Such a parametrisation introduces new degrees of freedom for the linearisation of the related frequency assignment map and plays an important role to the solvability of the output feedback pole assignment problem. The paper reviews the Global Asymptotic Linearisation method associated with the core versions of determinantal pole assignment problems and defines the conditions which characterises degenerate solutions of different types. Using the theory of ordered minimal bases, we provide a parametrisation of special families of degenerate compensators according to their degree. Finally, the special properties of degenerate solutions that allow frequency assignment are considered.



*Keywords:* Linear multivariable systems; output feedback control; algebraic methods; geometric methods; pole assignment.

## 1. INTRODUCTION

The construction of dynamic, or static, output feedback controllers that places the poles of a $p-$input, $m-$output, $n-$state multivariable system to arbitrary desired locations was always a challenging problem in control theory of linear systems. The pole assignment via output feedback is a multi-linear problem in the gain parameters and belongs to the family of Determinantal Assignment Problems (DAP) which has been introduced in (Karcanias and Giannakopoulos, 1984) as a framework that unifies the study of all pole-zero frequency assignment problems of determinantal type. The multi-linear nature of DAP has been handled by a blow-up type (algebro-geometric) methodology, which is known as Global Asymptotic Linearisation (Leventides and Karcanias, 1995, 1998) and uses as a basis the notion of degeneracy. Under certain conditions this methodology allows the computation of solutions of DAP. There are many challenging issues in the development of the DAP framework and amongst them are its ability to provide solutions even for non-generic cases, as well as providing approximate solutions (Leventides et al., 2014c),(Karcanias and Leventides, 2015) to the cases where generically there is no solution of the exact problem. The construction of such solutions requires the further development of the Global Linearisation methodology and an essential part of this is the parametrisation of the degenerate solutions. The notion of degeneracy has been first introduced by (Brockett and Byrnes, 1981) and characterise the boundary cases for the compensation or the feedback configuration. Such solutions have the significant property that linearise asymptotically the multilinear nature of the related frequency assignment map (i.e., the pole placement map in our case) and thus they become key instruments for developing Global Linearisation (Leventides and Karcanias, 1995, 1998). The classification of degenerate controllers and the parametrisation of such families play also an integral role for the application of the Global Linearisation methodology to special structure problems, such as decentralized control etc.

In (Leventides et al., 2014a,b) the authors have developed numerical techniques based on the Global Linearisation methodology that avoids the high sensitivity and the possible high gain nature of the compensation. In particular, they have used homotopy continuation techniques and modified Newton methods in order to be able to trace the solution fibre as far as possible from the locus of degenerate points and hence achieve solutions that assigns the desired pole polynomial with much better sensitivity properties regarding the closed-loop system.

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The current research is the algebraic part of the Global Asymptotic Linearisation methodology and its investigation is essential for the further development of the method. This paper extends the work which has started on the characterisation of the constant degenerate solutions in (Karcanias et al., 2013, 2014), based on the abstract DAP case. Here, an alternative approach on the original dynamic output feedback set-up is explored for the parametrisation of the degenerate solutions. This provides flexibility in selecting degenerate points that have full rank differential and thus enable frequency assignment. In the case where appropriate constant degenerate solutions do not exist we provide extensions in selecting dynamic degenerate solutions that enable the solvability of various control problems.

The paper is structured as follows: Section 2, defines the problem and reviews the background theory of minimal bases and the tools that will be needed in the sequel. The family of all degenerate solutions for a given multivariable system is defined and a classification of the degenerate gains into two different types (regular and irregular) is provided together with the necessary and sufficient conditions for their existence. Section 3, introduces the procedure for the construction of such compensators together with the parametrisation of the corresponding families of degenerate solutions according to their degree. Finally, in Section 4, the main ingredients of the Global Asymptotic Linearisation method are revisited and the sufficient conditions under which the differential of the related frequency assignment map has certain properties that allow arbitrary frequency assignment are given.

*Notation:* Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m \times n$ matrices over $\mathcal{F}$. If $H$ is a map, then $\mathcal{R}\{H\}$,$\mathcal{N}_r\{H\}$, $\mathcal{N}_l\{H\}$ denote the range, right, left null-spaces respectively and $\rho\{H\}$ its rank. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of $k$ integers from the set $\tilde{n} = \{1, 2, \dots, n\}$.

## 2. PROBLEM STATEMENT AND BACKGROUND TOOLS

### 2.1 Definition of the Problem

Let consider linear systems described by $S(A, B, C, D)$ state space descriptions with $n-$states, $p-$inputs and $m-$outputs, where $(A, B)$ is controllable, $(A, C)$ is observable, or by the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ where the rank is $\min(m, p)$. In terms of left, right coprime Matrix Fraction Descriptions (MFD), $G(s) \in \mathbb{R}^{m\times p}(s)$ may be represented as

$$G(s) = D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1}$$

or by the associated composite system matrix

$$M(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \in \mathbb{R}^{(p+m)\times p}[s] \tag{1}$$

where, $N_l(s)$, $N_r(s) \in \mathbb{R}^{m\times p}[s]$, $D_l(s) \in \mathbb{R}^{m\times m}[s]$ and $D_r(s) \in \mathbb{R}^{p\times p}[s]$. Throughout this paper we denote as

$$\mathcal{M} \triangleq colsp_{\mathbb{R}[s]}\{M(s)\}$$

the $\mathbb{R}[s]-$module which is uniquely defined by $G(s)$, it is a maximal Noetherian module (Forney, 1975) and its Forney dynamical indices are the controllability indices of any minimal realization $S(A, B, C)$ of $G(s)$. For the standard feedback control scheme and assuming a right coprime MFD representation for the system and left MFD for the controller the following frequency assignment problems are defined:

I) The *Dynamic Output Feedback* (DOF) pole assignment problem is expressed via

$$\det\{H(s)\cdot M(s)\} = \\ = \det\left\{[A_l(s), B_l(s)]\begin{bmatrix} D_r(s) \\ N_r(s)\end{bmatrix}\right\} = p(s)$$

II) The *Static Output Feedback* (SOF) pole assignment problem is defined as:

$$\det\{H\cdot M(s)\} = \det\left\{[I_p, K]\begin{bmatrix} D_r(s) \\ N_r(s)\end{bmatrix}\right\} = p(s)$$

where $p(s)$ denote the prime pole polynomial to be assigned. We may now examine some special feedback gains for which the well formed nature of the feedback configuration is lost, that is the degenerate gains. Let us first introduce the following notions:

*Definition 1.* Any matrix of the type

$$H(s) = [A(s), B(s)] \in \mathbb{R}^{p\times(p+m)}[s] \tag{2}$$

where, $A(s) \in \mathbb{R}^{p\times p}[s]$ and $rank\{H(s)\} = p$, will be a (dynamic) *generalized gain*.

□

*Remark 2.* A special family of *generalized gains* exists when $H(s) \equiv H$, i.e. a constant matrix of the type

$$H = [A, B] \in \mathbb{R}^{p\times(p+m)} \tag{3}$$

where, $A \in \mathbb{R}^{p\times p}$ and $rank\{H\} = p$. We define also, for any $K \in \mathbb{R}^{p\times m}$, as the composite gain of the feedback configuration, the matrix

$$\tilde{H} = [I_p, K] \in \mathbb{R}^{p\times(p+m)}$$

Generalized gains may be further classified into *regular* and *irregular* type. In particular, if $|A| \neq 0$, then the gain will be called *regular*, otherwise (i.e $|A| = 0$) it will be called *irregular*. It is well known that regular generalised gains correspond to standard bounded pre-compensation gains, given by $K = A^{-1}B$, whereas irregular generalised gains correspond to unbounded gains. We may now define:

*Definition 3.* Given a system described by the composite matrix $M(s) \in \mathbb{R}^{(m+p)\times p}[s]$ and for the standard feedback configuration, we say that the generalised gain, as in (2), is a *Generalised Degenerate Gain (GDG)* if

$$det\{H(s)\cdot M(s)\} \equiv 0 \tag{4}$$

□

*Remark 4.* A special family of GDGs is considered when $H(s) \equiv H \in \mathbb{R}^{p\times(p+m)}$ is a constant real matrix, that is the *constant Generalised Degenerate Gains* defined as

$$\det\{H\cdot M(s)\} = \det\{[A, B]\begin{bmatrix} D(s) \\ N(s)\end{bmatrix}\} \equiv 0 \tag{5}$$

We start off by investigating the conditions for existence of degenerate solutions.

*Proposition 5.* For the standard feedback configuration corresponding to a system described by $M(s)$, there exists