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Structural Stability and Equivalence of Linear 2D Discrete Systems ⋆

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Abstract: We study stability issues for linear two-dimensional (2D) discrete systems by means of the constructive algebraic analysis approach to linear systems theory. We provide a general definition of structural stability for linear 2D discrete systems which coincides with the existing definitions in the particular cases of the classical Roesser and Fornasini-Marchesini models. We then study the preservation of this structural stability by equivalence transformations. Finally, using the same framework, we consider the stabilization problem for equivalent linear systems.

Keywords: System theory, algebraic approaches, multidimensional systems, discrete systems, structural stability, stabilization methods

1. INTRODUCTION

The *algebraic analysis* or *behavioral* approach to linear systems theory is a unified mathematical framework to study multidimensional systems of linear functional equations appearing in control theory, engineering sciences, mathematical physics, ... See for instance [20, 25, 21, 6, 9, 23, 27, 14] and the references therein. A linear system can always be written as $R\,\eta = 0$, where $R \in D^{q\times p}$ is a $q \times p$ matrix with entries in a (noncommutative) polynomial ring D of functional operators and η is a vector of p unknown functions which belongs to a functional space. We then introduce the finitely presented left D-module $M := D^{1\times p}/(D^{1\times q}\,R)$. If $\mathcal{F}$ is a functional space having a left D-module structure, we consider the *linear system* or *behavior* $\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ and we have Malgrange's isomorphism $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$ (see [20]) which shows that system properties of $\ker_{\mathcal{F}}(R.)$ can be studied by means of module properties of M (and $\mathcal{F}$). Moreover, we nowadays have constructive algebraic techniques (e.g., constructive homological algebra using (noncommutative) Gröbner basis computations) and their implementations in several computer algebra systems at our disposal to study/check module properties of M. See for instance [6, 23] and the references therein.

The contribution of this paper consists in using the framework of the constructive algebraic analysis approach to linear systems theory to investigate stability and stabilization issues for linear 2D discrete systems. Note that the algebraic analysis approach has already been used to study stability and stabilization problems for linear multidimensional systems: see for instance [22] or more recently [26, 4] and the references therein. In the present work, we provide general definitions of *structural stability* and *stabilization* for linear 2D discrete systems which are coherent with the existing definitions in the particular cases of the classical Roesser and Fornasini-Marchesini models. Moreover, we take advantage of the results in [11, 12, 8] which constructively tackle the *equivalence problem* for linear systems in order to focus on how structural stability and stabilization properties are transmitted from a given linear system to an equivalent one. In particular, we study the impact of applying a state feedback control law to stabilize a system on an equivalent system. Finally, all our results are applied to the particular case of a generalized Fornasini-Marchesini model and its equivalent Roesser model (see [8]).

The paper is organized as follows. In Section 2, we recall some useful facts concerning Roesser and generalized Fornasini-Marchesini models and the equivalence of linear systems within the constructive algebraic analysis approach to linear systems theory. In Section 3, we introduce a general definition of structural stability for linear 2D discrete systems and we study its preservation via equivalence transformations. In Section 4, we consider stabilization issues in the same framework. Finally, in Section 5, we apply the previous results to the case of a generalized Fornasini-Marchesini model and its equivalent Roesser model.

Notation: We note $\mathbb{Q}$ (resp. $\mathbb{C}$) the field of rational (resp. complex) numbers. In the whole paper, $R \in D^{d_1\times d_2}$ means that R is a matrix with d_1 rows and d_2 columns whose entries are in (a ring) D and I_n denotes the identity matrix of dimension n. If $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extension of $\mathbb{C}$ in Alexandrov's sense, then we introduce the following two subsets of $\overline{\mathbb{C}}^2$: $\overline{\mathbb{S}}^2 := \{(z_1, z_2) \in \overline{\mathbb{C}}^2 \mid \forall i = 1, 2, |z_i| \geq 1\}$, and $\overline{\mathbb{D}}^2 := \{(z_1, z_2) \in \overline{\mathbb{C}}^2 \mid \forall i = 1, 2, |z_i| \leq 1\}$.

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2. PRELIMINARIES

2.1 Classical linear 2D discrete systems

In the present paper, we shall focus on linear 2D discrete systems, i.e., systems of linear equations whose dependent variables are discrete functions (sequences) of two independent variables denoted by i and j. In particular, we shall consider two classical explicit models of such systems which are the Roesser model [24] and the (generalized) Fornasini-Marchesini model [15]. Let us recall the particular form of these two models and how *structural stability* has been defined in both cases.

Roesser models: Roesser models have been introduced in [24]. They correspond to linear 2D discrete systems for which the equations are written under the (explicit) particular form:

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(i,j), \tag{1}$$

where x^h (resp. x^v) is the horizontal (resp. vertical) state vector of dimension d_h (resp. d_v), u is the input vector of dimension d_u, $A_{11} \in \mathbb{Q}^{d_h \times d_h}$, $A_{12} \in \mathbb{Q}^{d_h \times d_v}$, $A_{21} \in \mathbb{Q}^{d_v \times d_h}$, $A_{22} \in \mathbb{Q}^{d_v \times d_v}$, $B_1 \in \mathbb{Q}^{d_h \times d_u}$, $B_2 \in \mathbb{Q}^{d_v \times d_u}$. A notion of *structural stability* has been introduced for such particular models. See [19, 1, 3] and the references therein.

Definition 1. A Roesser model (1) is said to be *structurally stable* if

$$\forall (\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2, \ \det \begin{pmatrix} \lambda_1 I_{d_h} - A_{11} & -A_{12} \\ -A_{21} & \lambda_2 I_{d_v} - A_{22} \end{pmatrix} \neq 0. \tag{2}$$

If (1) is not structurally stable, then applying the state feedback control law

$$u(i,j) = (K_1 \quad K_2) \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}, \tag{3}$$

where $K_1 \in \mathbb{Q}^{d_u \times d_h}$ and $K_2 \in \mathbb{Q}^{d_u \times d_v}$ to (1), we obtain the new Roesser model

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \begin{pmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}.$$

If the latter model is structurally stable, then we say that the state feedback control law (3) *stabilizes* (1).

Fornasini models: These models take their origin in the work by Fornasini and Marchesini. See for instance [15] and the references therein. They were generalized by Kurek [18] and Kaczorek [16, 17]. In the present paper, we call *Fornasini model*, a linear 2D discrete system for which the equations are written under the (explicit) particular form:

$$\begin{aligned} x(i+1,j+1) &= F_1\, x(i+1,j) + F_2\, x(i,j+1) + F_3\, x(i,j) \\ &\quad + G_1\, u(i+1,j) + G_2\, u(i,j+1) + G_3\, u(i,j), \end{aligned} \tag{4}$$

where x is the state vector of dimension d_x, u is the input vector of dimension d_u, $F_1 \in \mathbb{Q}^{d_x \times d_x}$, $F_2 \in \mathbb{Q}^{d_x \times d_x}$, $F_3 \in \mathbb{Q}^{d_x \times d_x}$, $G_1 \in \mathbb{Q}^{d_x \times d_u}$, $G_2 \in \mathbb{Q}^{d_x \times d_u}$, $G_3 \in \mathbb{Q}^{d_x \times d_u}$. To our knowledge, a notion of structural stability for Fornasini models (4) has been defined only in the particular case $F_3 = G_3 = 0$. See [15, 19, 3] and the references therein.

Definition 2. A Fornasini model (4) with $F_3 = G_3 = 0$ is said to be *structurally stable* if

$$\forall (\lambda_1, \lambda_2) \in \overline{\mathbb{D}}^2, \ \det(I_{d_x} - \lambda_1 F_1 - \lambda_2 F_2) \neq 0. \tag{5}$$

If a Fornasini model (4) with $F_3 = G_3 = 0$ is not structurally stable, then applying the state feedback control law

$$u(i,j) = K\, x(i,j), \tag{6}$$

where $K \in \mathbb{Q}^{d_u \times d_x}$ to (4), we obtain the new Fornasini model

$$x(i+1,j+1) = (F_1 + G_1 K)\, x(i+1,j) + (F_2 + G_2 K)\, x(i,j+1).$$

If the latter model is structurally stable, then we say that the state feedback control law (6) *stabilizes* (4).

We have the following straightforward lemma:

Lemma 3. With the above notation, the condition (5) is equivalent to:

$$\forall (\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2, \ \det(\lambda_1 \lambda_2 I_{d_x} - \lambda_1 F_1 - \lambda_2 F_2) \neq 0. \tag{7}$$

Remark 4. Note that the conditions (2), (5), and (7) of structural stability can be effectively checked using the efficient algorithm recently developed in [5] and implemented in the computer algebra system Maple.

Let $D = \mathbb{Q}\langle \sigma_i, \sigma_j \rangle$ denote the commutative ring of partial (forward) shift operators with constant rational coefficients, i.e., for a bivariate sequence $f(i,j)$, we have $\sigma_i f(i,j) = f(i+1,j)$, $\sigma_j f(i,j) = f(i,j+1)$, and we further have $\sigma_i \sigma_j = \sigma_j \sigma_i$, where $\sigma_i \sigma_j$ stands for the composition of operators $\sigma_i \circ \sigma_j$. An operator $P \in D$ can be written as $P = \sum_{m,l} p_{ml}\, \sigma_i^m\, \sigma_j^l$, where $p_{ml} \in \mathbb{Q}$, the sum is finite, and, for a bivariate sequence $f(i,j)$, we thus have $P\, f(i,j) = \sum_{m,l} p_{ml}\, f(i+m, j+l)$. Within the algebraic analysis approach to linear systems theory:

(1) The Roesser model (1) is written as $R\,\eta = 0$, where $R \in D^{(d_h+d_v) \times (d_h+d_v+d_u)}$ and η are defined by

$$R = \begin{pmatrix} I_{d_h}\, \sigma_i - A_{11} & -A_{12} & -B_1 \\ -A_{21} & I_{d_v}\, \sigma_j - A_{22} & -B_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} x^h \\ x^v \\ u \end{pmatrix}.$$

It is then studied by means of the factor D-module $M = D^{1 \times (d_h+d_v+d_u)}/(D^{1 \times (d_h+d_v)}\, R)$,

(2) The Fornasini model (4) is written as $R\,\eta = 0$, where $R \in D^{d_x \times (d_x+d_u)}$ is defined by

$$R = (I_{d_x} \sigma_i \sigma_j - F_1 \sigma_i - F_2 \sigma_j - F_3 \ \ -G_1 \sigma_i - G_2 \sigma_j - G_3),$$

and $\eta = \left(x^T \ \ u^T\right)^T$. It is then studied by means of the D-module $M = D^{1 \times (d_x+d_u)}/(D^{1 \times d_x}\, R)$.

2.2 Equivalence in the framework of algebraic analysis

Using the framework of the algebraic analysis approach to linear systems theory recalled in the introduction, equivalent linear systems correspond to isomorphic left D-modules and the *equivalence problem* has been constructively studied in the recent works [11, 12, 8]. Let us summarize part of the results obtained in the latter works:

Lemma 5. Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ and consider the associated left D-modules $M = D^{1 \times p}/(D^{1 \times q}\, R)$ and $M' = D^{1 \times p'}/(D^{1 \times q'}\, R')$.

(1) The existence of a homomorphism $f \in \hom_D(M, M')$ is equivalent to the existence of $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the identity

$$R\, P = Q\, R'. \tag{8}$$