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New Architectures for Hierarchical Predictive Control

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Abstract: We analyze the structure of the Euler-Lagrange (EL) conditions of a long-horizon optimal control problem. The analysis reveals that the conditions can be solved by using block Gauss-Seidel (GS) schemes. We prove that such schemes can be implemented in the primal space by solving sequences of short-horizon optimal control problems. This analysis also reveals that a traditional receding-horizon (RH) scheme is equivalent to performing a single GS sweep. We have also found that we can use adjoint information from a coarse long-horizon problem to construct terminal penalties that correct the policies of the RH scheme. We observe that this scheme can be interpreted as a hierarchical controller in which a coarse high-level controller transfers long-horizon information to a low-level, short-horizon controller of fine resolution. The results open the door to a new family of hierarchical control architectures that can handle multiple time scales systematically.

Keywords: RH, hierarchical control, time scales, Euler-Lagrange

1. BASIC NOTATION AND SETTING

We start by providing basic notation and defining the technical problem. Relevant references are provided as we proceed with the discussion. We consider the following *long-horizon* optimal control problem:

$$\min_{z(\cdot),u(\cdot)} \quad \int_0^T \varphi(z(\tau),u(\tau),w(\tau))d\tau \tag{1a}$$
$$\text{s.t.} \quad \dot{z}(\tau) = f(z(\tau),u(\tau),w(\tau)),\ \tau \in [0,T] \tag{1b}$$
$$z(0) = \bar{z}. \tag{1c}$$

Here, $z(\cdot)$, $u(\cdot)$, and $w(\cdot)$ are state, control, and disturbance trajectories, respectively. The cost and system mappings $\varphi(\cdot)$ and $f(\cdot)$ are assumed to be smooth.

We *lift* the long-horizon problem by partitioning the horizon T into n stages. This lifting approach was proposed by Bock and Plitt (1984) in the context of multiple-shooting. We define the sets $\mathcal{N} := \{0..n-1\}$ and $\mathcal{N}^- := \mathcal{N} \setminus \{n-1\}$; and we assume the stages to be of equal length $h := T/n$. The partitioning gives rise to the *lifted* problem,

$$\min_{z_k(\cdot),u_k(\cdot)} \quad \sum_{k\in\mathcal{N}} \int_0^h \varphi(z_k(\tau),u_k(\tau),w_k(\tau))d\tau \tag{2a}$$
$$\text{s.t.}\ \dot{z}_k(\tau) = f(z_k(\tau),u_k(\tau),w_k(\tau)),\ k\in\mathcal{N},\ \tau \in [0,h] \tag{2b}$$
$$z_{k+1}(0) = z_k(h),\ k \in \mathcal{N}^- \tag{2c}$$
$$z_0(0) = \bar{z}. \tag{2d}$$

We will analyze the stage structure of the lifted optimal control problem. In doing so we will reduce the notation to a minimum, in such a way that it retains the essential features of the structure we are interested in highlighting. We first note that we do not consider inequality and path constraints and we eliminate dependencies of the mappings on the disturbances. These changes will not alter the stage structure of the lifted problem. We transcribe the lifted problem into a finite-dimensional nonlinear programming problem by applying an implicit Euler scheme with m inner stages of equal length $\delta := h/m$ (other discretization schemes can also be applied). We define the sets of inner discretization points $\mathcal{M} := \{0..m-1\}$. The discretized problem is,

$$\min_{z_{k,j},u_{k,j}} \quad \sum_{k\in\mathcal{N}} \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1},u_{k,j+1}) \tag{3a}$$
s.t.
$$(\nu_{k,j+1})\ z_{k,j+1} = z_{k,j} + \delta f(z_{k,j+1},u_{k,j+1}),\ k\in\mathcal{N}, j\in\mathcal{M} \tag{3b}$$
$$(\lambda_k) \qquad z_{k,0} = z_{k-1,m},\ k\in\mathcal{N}. \tag{3c}$$

Here, $\nu_{k,j}$ are the dual variables of the inner dynamic equations (3b), and λ_k are the dual variables of the stage-transition equations (4b). The dual variables are scaled by the constant $1/\delta$. We use the dummy parameter $z_{-1,m} := \bar{z}$ to simplify notation. We denote the discretized long-horizon problem (4) as $\mathcal{P}$. We simplify notation further by eliminating the dynamic equations from the notation (3b). This, again, does not alter the stage structure. We obtain the compact problem,

$$\min_{z_{k,j},u_{k,j}} \quad \sum_{k\in\mathcal{N}} \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1},u_{k,j+1}) \tag{4a}$$
$$\text{s.t.} \quad (\lambda_k)\ \ z_{k,0} = z_{k-1,m},\ k\in\mathcal{N}. \tag{4b}$$

A controller based on recursive solutions of $\mathcal{P}$ must capture disturbance signals that evolve over multiple time scales (e.g., noise, weather, prices) and must handle slow and fast components of the dynamical system (e.g., fast and slow chemical reactions, recycle systems). Despite advances in computational methods for optimal control, this might not be possible to do. This is is because the solution of $\mathcal{P}$ might require very fine discretization meshes and/or

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expensive numerical integration procedures to capture dynamic effects at all time scales. Reviews on the topic are presented by Diehl et al. (2009) and Zavala and Biegler (2009). We also note that the presence of multiple time scales plays a role in the resolution and update frequency of the control. For instance, as noted by Findeisen et al. (2007), if disturbances are fast it is necessary to use a compatible control resolution.

The complexity of $\mathcal{P}$ is traditionally addressed by using a RH scheme which seeks to approximate the optimal long-horizon policy by solving sequences of fine-resolution short-horizon problems. In particular, one can solve the following short-horizon problems sequentially for $k = 0, ..., N-1$:

$$\min_{z_{k,j},u_{k,j}} \quad \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) \tag{5a}$$

$$\text{s.t.} \quad (\lambda_k) \quad z_{k,0} = z_{k-1,m}. \tag{5b}$$

Here, the initial state $z_{k-1,m}$ is fixed and is obtained from the solution of the problem at $k-1$. We will show that this RH scheme is a block GS iteration applied to the solution of the Euler-Lagrange (EL) conditions of (4). This observation will help us derive hierarchical schemes to address the intractability of $\mathcal{P}$.

2. STRUCTURE OF EULER-LAGRANGE CONDITIONS

We group variables by stages by defining the vectors $\mathbf{z}_k := (z_{k,0}, ..., z_{k,m})$, $\mathbf{u}_k := (u_{k,1}, ..., u_{k,m})$, and $\nu_k := (\nu_{k,1}, ..., \nu_{k,m})$. We thus obtain the block form of $\mathcal{P}$,

$$\min_{\mathbf{u}_k} \quad \sum_{k\in\mathcal{N}} \phi(\mathbf{z}_k, \mathbf{u}_k) \tag{6a}$$

$$\text{s.t.} \quad (\lambda_k) \quad \overline{\Pi}_k \mathbf{z}_k = \underline{\Pi}_k\, \mathbf{z}_{k-1}, \ k \in \mathcal{N}. \tag{6b}$$

The structure of the mapping $\phi(\cdot)$ is given by:

$$\phi(\mathbf{z}_k, \mathbf{u}_k) := \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}). \tag{7}$$

The coefficient matrices $\overline{\Pi}_k$ and $\underline{\Pi}_k$ satisfy $\overline{\Pi}_k \mathbf{z}_k = z_{k,0}$ and $\underline{\Pi}_k \mathbf{z}_{k-1} = z_{k-1,m}$. We also define the fixed dummy vector $\mathbf{z}_{-1}$ satisfying $\underline{\Pi}_0\, \mathbf{z}_{-1} = z_{-1,m} = \bar{z}$.

The Lagrange function of $\mathcal{P}$ is given by

$$\mathcal{L}(\mathbf{z}_k, \mathbf{u}_k, \lambda_k) := \sum_{k\in\mathcal{N}} \phi(\mathbf{z}_k, \mathbf{u}_k) - \lambda_k^T (\overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1}), \tag{8}$$

and its first-order optimality conditions are

$$0 = \nabla_z \phi_k - \overline{\Pi}_k^T \lambda_k + \underline{\Pi}_{k+1}^T \lambda_{k+1}, \ k \in \mathcal{N}^- \tag{9a}$$

$$0 = \nabla_z \phi_{n-1} - \overline{\Pi}_{n-1}^T \lambda_{n-1} \tag{9b}$$

$$0 = \nabla_u \phi_k, \ k \in \mathcal{N} \tag{9c}$$

$$0 = \overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1}, \ k \in \mathcal{N}. \tag{9d}$$

Here, $\nabla_z \phi_k := \nabla_{\mathbf{z}_k} \phi(\cdot)$ and $\nabla_u \phi_k := \nabla_{\mathbf{u}_k} \phi(\cdot)$. System (9) is the discrete-time version of the EL conditions of the lifted problem (2). Moreover, the dual variables λ_k can be tied together to form discrete-time profiles of the adjoint variables of the lifted problem. These properties are discussed in the book of Biegler (2010).

We note that the block component of the EL conditions corresponding to each stage $k \in \mathcal{N}^-$ is given by,

$$0 = \nabla_z \phi_k - \overline{\Pi}_k^T \lambda_k + \underline{\Pi}_{k+1}^T \lambda_{k+1} = 0 \tag{10a}$$

$$0 = \nabla_u \phi_k \tag{10b}$$

$$0 = \overline{\Pi}_k \mathbf{z}_k - \underline{\Pi}_k \mathbf{z}_{k-1} \tag{10c}$$

For fixed $\underline{\Pi}_k \mathbf{z}_{k-1} = z_{k-1,m}$ and λ_{k+1}, (10) are the first-order conditions of the primal stage problem:

$$\min_{\mathbf{z}_k,\mathbf{u}_k} \quad \phi(\mathbf{z}_k, \mathbf{u}_k) + (\lambda_{k+1})^T \underline{\Pi}_{k+1} \mathbf{z}_k \tag{11a}$$

$$\text{s.t.} \quad (\lambda_k) \quad \overline{\Pi}_k \mathbf{z}_k = \underline{\Pi}_k\, \mathbf{z}_{k-1}. \tag{11b}$$

For the last stage $k = n-1$ we have the block component of the EL conditions:

$$0 = \nabla_z \phi_{n-1} - \overline{\Pi}_{n-1}^T \lambda_{n-1} = 0 \tag{12a}$$

$$0 = \nabla_u \phi_{n-1} \tag{12b}$$

$$0 = \overline{\Pi}_{n-1} \mathbf{z}_{n-1} - \underline{\Pi}_{n-1} \mathbf{z}_{n-2}. \tag{12c}$$

For fixed $\underline{\Pi}_{n-1} \mathbf{z}_{n-2} = z_{n-2,m}$ these are the first-order conditions of the primal stage problem,

$$\min_{\mathbf{z}_{n-1},\mathbf{u}_{n-1}} \quad \phi(\mathbf{z}_{n-1}, \mathbf{u}_{n-1}) \tag{13a}$$

$$\text{s.t.} \quad (\lambda_{n-1}) \quad \overline{\Pi}_{n-1} \mathbf{z}_{n-1} = \underline{\Pi}_{n-1}\, \mathbf{z}_{n-2}. \tag{13b}$$

From the structure of (10) and (11) we can see that coupling between neighboring stages $k-1$, k, and $k+1$ is introduced through the states $z_{k-1,m}$ and adjoints λ_{k+1}.

3. BLOCK GS SCHEMES

Our key observation is that we make is that we can solve the EL conditions (9) of the long-horizon problem by using *block* GS schemes. Assume that the adjoints λ_k are fixed to $\lambda_k^\ell = 0$ for all $k \in \mathcal{N}$. At stage $k = 0$ and with fixed $z_{-1}^\ell = \bar{z}$ we solve the short-horizon problem:

$$\min_{z_{k,j},u_{k,j}} \quad \sum_{j\in\mathcal{M}} \varphi(z_{k,j+1}, u_{k,j+1}) + \delta(\lambda_{k+1}^\ell)^T z_{k,m} \tag{14a}$$

$$\text{s.t.} \quad (\lambda_k) \quad z_{k,0} = z_{k-1,m}^\ell. \tag{14b}$$

We refer to this problem as $\mathcal{P}_k$ and introduce the notation

$$(z_{k,m}^{\ell+1}, \lambda_k^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k-1,m}^\ell, \lambda_{k+1}^\ell) \tag{15}$$

to indicate the inputs and outputs of problem $\mathcal{P}_k$. The primal-dual solution of $\mathcal{P}_k$ solves block k of the EL conditions (10) for fixed initial state $\underline{\Pi}_k \mathbf{z}_{k-1} = z_{k-1,m}^\ell$ and adjoint $\lambda_{k+1} = \lambda_{k+1}^\ell$. Note also that $\mathcal{P}_k$ is equivalent to the stage problem (11).

From the solution of $\mathcal{P}_k$ we obtain the terminal state $z_{k,m}^{\ell+1}$ and we use this as initial state for $\mathcal{P}_{k+1}$ to compute $(z_{k+1,n}^{\ell+1}, \lambda_{k+1}^{\ell+1}) \leftarrow \mathcal{P}_k(z_{k,m}^{\ell+1}, \lambda_{k+2}^\ell)$. We continue the recursion until reaching the last stage, $k = n-1$. At this stage we solve problem $\mathcal{P}_{n-1}$:

$$\min_{z_{n-1,j},u_{n-1,j}} \quad \sum_{j\in\mathcal{M}} \varphi(z_{n-1,j+1}, u_{n-1,j+1}) \tag{16a}$$

$$\text{s.t.} \quad (\lambda_{n-1}) \quad z_{n-1,0} = z_{n-2,m}^\ell. \tag{16b}$$

With this we compute $(z_{n-1,m}^{\ell+1}, \lambda_{n-1}^{\ell+1}) \leftarrow \mathcal{P}_{n-1}(z_{n-2,m}^\ell, 0)$. The primal-dual solution of $\mathcal{P}_{n-1}$ solves the optimality system (12) for fixed initial state $\underline{\Pi}_{n-1} \mathbf{z}_{n-2} = z_{n-2,m}^\ell$ obtained from the solution of $\mathcal{P}_{n-2}$. Moreover, $\mathcal{P}_{n-1}$ is equivalent to (13).

After solving $\mathcal{P}_{n-1}$ we have updated all the state (primal) $z_k^{\ell+1}$ and adjoint $\lambda_k^{\ell+1}$ variables. In Figure 1 we can see