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Local stability of discrete-time TS fuzzy systems ⋆

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Abstract: This paper considers local stability analysis of discrete-time Takagi-Sugeno fuzzy systems, for which classically in the TS literature only global stability is considered. Using a common quadratic and nonquadratic Lyapunov function, respectively, LMI conditions are developed to establish local stability of an equilibrium point. An estimate of the region of attraction of this point is also determined. The developed conditions are illustrated on a numerical example.

Keywords: discrete-time fuzzy system; local stability; Lyapunov stability

1. INTRODUCTION

Takagi-Sugeno (TS) fuzzy models (Takagi and Sugeno, 1985) are nonlinear systems represented as convex combinations of local linear models on a compact set of the state-space. Such a representation usually facilitates the automatic stability analysis or controller design of the nonlinear system.

For the analysis of TS models the direct Lyapunov approach has been used. Using initially quadratic Lyapunov functions (Tanaka et al., 1998; Tanaka and Wang, 2001; Sala et al., 2005), later on piecewise continuous Lyapunov functions (Johansson et al., 1999; Feng, 2004), and more recently, nonquadratic Lyapunov functions (Guerra and Vermeiren, 2004; Kruszewski et al., 2008; Mozelli et al., 2009), in general linear matrix inequality (LMI) conditions are developed, which can be solved using available convex optimization methods.

The initial results, in particular those involving common quadratic Lyapunov functions, develop conditions that, when satisfied, imply the global stability of the TS model. This in fact means that any trajectory starting in the largest Lyapunov level set included in the considered compact set of the state-space will converge. In the case of the continuous-time TS models, with the introduction of nonquadratic Lyapunov functions, the developments involve the derivatives of the membership functions. Due to this, local stability results have been obtained, with the domain given by the bounds on the derivatives (Tanaka et al., 2003; Guerra and Bernal, 2009; Mozelli et al., 2009), usually being translated into bounds on the states.

In the discrete-time case, since the variation of the Lyapunov function does not involve any derivatives and thus further conditions, non-quadratic Lyapunov functions have shown a real improvement (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011; Lendek et al., 2013, 2015) for developing global stability and design conditions. It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework. More recently, by using Polya's theorem (Montagner et al., 2007; Sala and Ariño, 2007) asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability in the sense of a chosen quadratic or nonquadratic Lyapunov function. Ding (2010) gave ANS stability conditions for both membership function-dependent model and membership function-dependent Lyapunov matrix. By increasing the complexity of the homogeneously polynomially parameter-dependent Lyapunov functions, in theory any sufficiently smooth Lyapunov function can be approximated. Unfortunately, the number of LMIs that have to be solved increase quickly, leading to numerical intractability (Zou and Yu, 2014). However, all these results involve global stability, i.e., if an equilibrium point is not globally stable, no conclusion can be drawn.

With the considerations above, in this paper, we consider the problem of establishing local stability of discrete-time TS models and estimating a domain of attraction of the equilibrium point. The structure of the paper is as follows. Section 2 presents the notations used in this paper and motivates our work through a simple example. Section 3 develops the proposed conditions for stability

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analysis, using a common quadratic and a nonquadratic Lyapunov function, respectively. The developed conditions are discussed and illustrated on a numerical example. Section 4 concludes the paper.

2. NOTATION AND PRELIMINARIES

In this paper we develop sufficient conditions for the local stability of nonlinear discrete-time systems represented by Takagi-Sugeno (TS) fuzzy models. Thus, we consider systems of the form

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))A_i\boldsymbol{x}(k)\\ &= A_z\boldsymbol{x}(k)\end{aligned} \tag{1}$$

where $\boldsymbol{x}$ denotes the state vector, r is the number of rules, $\boldsymbol{z}$ is the scheduling vector, h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, and A_i, $i = 1, 2, \ldots, r$ are the local models. To motivate the research presented hereafter, consider the following example.

Example 1. Consider the nonlinear system:

$$\begin{aligned}x_1(k+1) &= x_1^2(k)\\ x_2(k+1) &= x_1(k) + 0.5x_2(k)\end{aligned} \tag{2}$$

with $x_1(k) \in [-a, a]$, $a > 0$ being a parameter. It can be easily seen that (2) is locally stable for $x_1 \in (-1, 1)$.

The nonlinearity is x_1^2 and using the sector nonlinearity approach (Ohtake et al., 2001) on the domain $x_1(k) \in [-a, a]$, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x}$$

with $h_1(x_1) = \frac{a-x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix}-a & 0\\ 1 & 0.5\end{pmatrix}$, $A_2 = \begin{pmatrix}a & 0\\ 1 & 0.5\end{pmatrix}$.

If $a < 1$, e.g., $a = 0.9$, the stability of the TS model can be easily proven e.g., using a common quadratic Lyapunov function.

If the sector nonlinearity approach is applied for $a > 1$, without including further conditions, no conclusion can be drawn regarding the stability of the TS model. A condition that leads to the feasibility of the associated LMI problem and thus makes it possible to draw some conclusion of local stability is e.g., $x_1^2(k) \geq 0.9x_1^2(k+1)$. However, the question on how to obtain such a condition and its exact interpretation remains open. □

In what follows, 0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry. The subscript $z+m$ (as in A_{z+m}) stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., at $\boldsymbol{z}(k+m)$. We will also make use of the following results:

Lemma 2. (Skelton et al., 1998) Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{R} \in \mathbb{R}^{m \times n_x}$ such that $\operatorname{rank}(\mathcal{R}) < n_x$. The two following expressions are equivalent:

(1) $\boldsymbol{x}^TQ\boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, \mathcal{R}\boldsymbol{x} = 0\}$
(2) $\exists \mathcal{M} \in \mathbb{R}^{m \times n_x}$ such that $Q + \mathcal{M}\mathcal{R} + \mathcal{R}^T\mathcal{M}^T < 0$ □

Lemma 3. (S-procedure) Consider matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, such that $\boldsymbol{x}^TF_i\boldsymbol{x} \geq 0$, $i = 1, \ldots, p$, and the quadratic inequality condition

$$\boldsymbol{x}^TF_0\boldsymbol{x} > 0 \tag{3}$$

$\boldsymbol{x} \neq 0$. A sufficient condition for (3) to hold is: there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, such that

$$F_0 - \sum_{i=1}^{p} \tau_iF_i > 0$$

. □

Analysis and design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T\sum_{j=1}^{r}\sum_{k=1}^{r} h_j(\boldsymbol{z}(k))h_k(\boldsymbol{z}(k))\Gamma_{j,k}\boldsymbol{x} < 0 \tag{4}$$

where $\Gamma_{j,k}$, $j, k = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 4. (Wang et al., 1996) The double-sum (4) is negative, if

$$\begin{aligned}&\Gamma_{ii} < 0\\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, i < j\end{aligned} \tag{5}$$

□

Lemma 5. (Tuan et al., 2001) The double-sum (4) is negative, if

$$\begin{aligned}&\Gamma_{ii} < 0\\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, i \neq j\end{aligned} \tag{6}$$

□

3. LOCAL STABILITY ANALYSIS

Consider the TS model (1), repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_z\boldsymbol{x}(k) \tag{7}$$

defined on the domain $\mathcal{D}$ including the origin.

Our goal is to develop conditions for this system to have a locally asymptotically stable equilibrium point in $\boldsymbol{x} = 0$ and determine a region of attraction. For this, let us first assume that

Assumption 6. There exists a domain $\mathcal{D}_R \subset \mathcal{D}$ and a symmetric matrix $R = R^T$ so that

$$\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T R \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathcal{D}_R$. □

Note that this assumption can always be satisfied, e.g., by reducing $\mathcal{D}_R$ to the origin.

3.1 Local quadratic stability

The following result is straightforward.

Theorem 7. The discrete-time nonlinear model (1) is locally asymptotically stable if there exist matrices $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ and scalar $\tau > 0$ so that

$$\begin{pmatrix}-P & (*)\\ M_zA_z & P - M_z - M_z^T\end{pmatrix} + \tau R < 0 \tag{8}$$

Moreover, the region of attraction , i.e., the region from which all trajectories converge to zero, includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$. □