Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 49-5 (2016) 049–054

Two stochastic filters and their interval extensions

Tran Tuan Anh * **, Françoise Le Gall** * **, Carine Jauberthie** *
Louise Travé-Massuyès *

* *LAAS-CNRS, Université de Toulouse, CNRS, UPS, Toulouse, France*

Abstract: In this paper, two well-known stochastic filtering algorithms, Kalman and particle filters, are presented. Their extensions using interval analysis are described. A fuel cell system case study is considered together with specific scenarios representing situations in which interval filters are relevant. The results confirm the advantage of the interval filters in such situations.

Keywords: State estimation, Kalman filtering, particle filtering, stochastic systems, bounded error, Interval analysis

1. INTRODUCTION

Filtering problem has a crucial role in many engineering applications where state estimation is necessary to design the control laws or monitor system performance. Many approaches used to design filters have been developed and can be classified according to the modeling of uncertainties. Some of them take the statistical modeling of uncertainties into account, e.g. Kalman filter is subject to Gaussian noises. The general idea of stochastic filter is to form an approximation for the real probability distribution *a posteriori* of real state given the noisy measurements of the system. The second group of methods relies on the representation of uncertainties by compact sets (intervals, zonotopes), i.e. no assumption about the statistical properties is then required. This group of methods is known as set-membership approach. Set-membership state estimation can be based on interval analysis that was introduced by [Moore (1966)]. Several algorithms have been proposed, in particular interval Kalman filter (IKF) developed by [Chen et al. (1997)] and its improvement [Xiong et al. (2013)], box particle filter (BPF) [Gning et al. (2007)] which are the extensions of stochastic filters in bounded-error context. Other set-membership approaches are dedicated to linear models, including ellipsoid shaped methods [Milanese and Novara (2002)], parallelotope and zonotope based methods [Combastel (2005)] or a combination of Kalman filter and zonotope based approach [Combastel (2015)]. The result of set-membership approaches is a compact set (intervals, zonotopes) in the state space containing the values of all the states that are consistent both with the uncertain model and the measurement. The present paper focuses on a comparative study between stochastic filters (Kalman filter, particle filter) and set-membership approach based on interval analysis.

The paper is organized as follows: after developing the problem formulation in Section 2, the main ideas of interval analysis are presented in Section 3. The stochastic filters (Kalman and particle filter) and their box extensions are detailed in Sections 4 and 5. The advantages and drawbacks of the considered filtering algorithms are illustrated in Section 6 by a state estimation problem for a fuel cell system.

2. PROBLEM FORMULATION

Consider the following nonlinear system:

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \\ \mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{v}_k). \end{cases} \tag{1}$$

where n_x, n_u, n_w, n_y, and n_v are respectively the dimensions of the state $\mathbf{x}$, input $\mathbf{u}$, process noise $\mathbf{w}$, measurement $\mathbf{y}$ and measurement error $\mathbf{v}$ vectors. The functions $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$, and $\mathbf{h} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_y}$ are possibly nonlinear functions. In this paper, the process noises and measurement errors can be represented by a statistical model or by an interval containing all possible values of uncertainties.

In the case of a linear system with Gaussian noises, the conventional Kalman filter is optimal with small computational effort. However, if there exist bounded parameter uncertainties due to modeling error, the Kalman filter cannot guarantee a good performance for the state estimation problem. A solution is to apply the improved interval Kalman filter in which the noises are modeled in a stochastic framework but parameter uncertainties are assumed to be bounded. For instance, we consider the following linear system:

$$\begin{cases} \mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k\mathbf{x}_k + \mathbf{v}_k, \end{cases} \tag{2}$$

where the matrices A_k, B_k, C_k and the covariance matrices Q_k, R_k of process noise w_k and measurement error v_k are assumed unknown and bounded by known interval matrices, as defined in the following section.

3. INTERVAL ANALYSIS

This section introduces some basic notions of interval analysis that are useful to deal with bounded uncertainties [Moore (1966)] and [Jaulin et al. (2001)].

A real interval, denoted $[x]$ is defined as a closed and connected subset of $\mathbb{R}$:

$$[x] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}, \tag{3}$$

where $\underline{x}$ and $\overline{x}$ are respectively the lower and upper bound of interval $[x]$. The width of an interval $[x]$ is defined by $|[x]| = \overline{x} - \underline{x}$, and its center is $C([x]) = (\overline{x} + \underline{x})/2$. The set of real intervals is denoted by $\mathbb{IR}$.

A box of $\mathbb{IR}^n$ is defined as a Cartesian product of n intervals:

$$[\mathbf{x}] = [x_1] \times [x_2] \times \cdots \times [x_n]. \tag{4}$$

10.1016/j.ifacol.2016.07.088

The set of $m \times n$ real interval matrices is denoted by $\mathbb{IR}^{m\times n}$. Classical operations for intervals, interval vectors, interval matrices (e.g. $+,-,\times,\div,\cap,\cup,\subset,\supset$) are extensions of the same operations for reals, real vectors, real matrices. Once the basic notions of interval analysis are defined, it is possible to evaluate a vectorial function $\mathbf{f}$ of interval variables involving a finite number of arithmetic operations as $[\mathbf{f}]$. The interval function $[\mathbf{f}]$ from $\mathbb{IR}^n$ to $\mathbb{IR}^m$ is an inclusion function for $\mathbf{f}$ if: $\mathbf{f}([\mathbf{x}]) \subseteq [\mathbf{f}]([\mathbf{x}]), \forall [\mathbf{x}] \subseteq \mathbb{IR}^n$. An inclusion function is convergent if, for any sequence of boxes $[\mathbf{x}(k)]$:

$$\lim_{k\to\infty} \omega([\mathbf{x}(k)]) = 0 \Rightarrow \lim_{k\to\infty} \omega([\mathbf{f}]([\mathbf{x}(k)])) = 0, \quad (5)$$

where $\omega([\mathbf{x}])$ is the width of the box $[\mathbf{x}]$ defined as the maximum of the widths of its interval components. Finding inclusion functions evaluating in reasonable computational time and as close as possible to the image of $\mathbf{f}([\mathbf{x}])$ is one of the main objective of interval analysis.

Interval analysis is also a means of solving systems of equations given bounded initial conditions. Consider a system of equations:

$$\mathbf{f}(\mathbf{x}) = 0, \quad (6)$$

where $\mathbf{x}$ is a vector of n variables $x_1, x_2, \ldots, x_n$ connected by m constraints $\mathbf{f} = (f_1, f_2, \ldots, f_m)$. Given the initial domain $[\mathbf{x}]$ of variables, the goal is to compute the smallest box $[\mathbf{x}'] \subset [\mathbf{x}]$ containing all solutions $\mathbf{x}$ of (6). This problem can be formulated as a *Constraint Satisfaction Problem* (CSP), denoted as $\mathcal{H}$:

$$\mathcal{H} : (\mathbf{f}(\mathbf{x}) = 0, \mathbf{x} \in [\mathbf{x}]). \quad (7)$$

Solving this CSP implies finding the smallest box $[\mathbf{x}'] \subset [\mathbf{x}]$ enclosing the solution set $\mathbf{S}$ of $\mathcal{H}$:

$$\mathbf{S} = \{\mathbf{x} \in [\mathbf{x}] \,|\, \mathbf{f}(\mathbf{x}) = 0\}. \quad (8)$$

Any operator that can be used to replace $[\mathbf{x}]$ by a smaller domain $[\mathbf{x}']$ such that $\mathbf{S} \subseteq [\mathbf{x}'] \subseteq [\mathbf{x}]$ is called a *contractor* for $\mathcal{H}$. Several contractor algorithms are presented in [Jaulin et al. (2001)], including the Gauss elimination algorithm, the Gauss-Seidel algorithm, and linear programming.

In many applications, we must find the set of possible values of variable $\mathbf{x}$ given a possibly nonlinear function $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ and a subset $\mathbf{Y}$ of $\mathbb{R}^m$ such that $\mathbf{f}(\mathbf{x})$ belongs to $\mathbf{Y}$. This problem is known as a *Set inversion problem* defined as:

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) \in \mathbf{Y}\} = \mathbf{f}^{-1}(\mathbf{Y}). \quad (9)$$

For any $\mathbf{Y} \subseteq \mathbb{R}^m$ and for any function $\mathbf{f}$ admitting a convergent inclusion function $[\mathbf{f}]$, we can obtain two regular subpavings $\underline{\mathbf{X}}$ and $\overline{\mathbf{X}}$ (not necessarily box-shaped) containing the set $\mathbf{X}$ by using the algorithm SIVIA (Set Inverter Via Interval Analysis) [Jaulin and Walter (1993)].

4. KALMAN FILTER AND BOX EXTENSION

4.1 Kalman filter

The Kalman filter algorithm was originally developed for systems assumed to be represented with a linear state-space model:

$$\begin{cases} \mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k + \mathbf{w}_k, \\ \mathbf{y}_k = C_k\mathbf{x}_k + \mathbf{v}_k. \end{cases} \quad (10)$$

where A_k, B_k, and C_k are matrices of appropriate size. It is also assumed that this system is excited by uncorrelated zero mean Gaussian noise processes v_k and w_k with covariance Q_k and R_k respectively.

The Kalman filter algorithm contains two steps for each iteration: prediction and update (see [Kalman (1960)]).

4.2 Box Kalman filter

In Chen et al. (1997), the interval Kalman filter based on interval conditional expectation for interval linear systems has been developed. The matrices A_k, B_k, C_k of the linear system (10) are interval matrices, denoted by $[A_k]$, $[B_k]$, $[C_k]$. The initial condition $(\mathbf{x}_0, P_0)$, the control $\mathbf{u}_k$ and the measurements $\mathbf{y}_k$ could be boxes accounting for bounded uncertainties. The interval Kalman filter has the same structure as the conventional Kalman filter algorithm while preserving the statistical optimality and the recursive computational scheme.The authors propose to bypass the singularity problems with interval matrix inversion by using the upper bound of the interval matrix to be inverted. This leads to a sub-optimal solution that may not include all the real solutions consistent with the bounded uncertainties represented in the system.

With the advances in interval analysis and constraint propagation mentioned in Section 3, a new recursive estimator, known as *improved Interval Kalman Filter* (iIKF) has been proposed in Xiong et al. (2013). The iIKF provides the envelopes of the set of all possible optimal state estimations and covariance matrices of the Kalman filtering problem. Notice that the matrices $[P_k^-], [P_k^+]$ and $[S_k]$ should be positive definite, where $[P_k^-] = [A_k][P_{k-1}^+][A_k]^T + [Q_k]$, $[S_k] = [C_k][P_k^-][C_k]^T + [R_k]$, and $[P_k^+] = (I_{n_x} - [K_k][C_k])[P_k^-]$. A property of positive definite matrices is used to contract these matrices: all elements on the diagonal are positive. This property is guaranteed by three CSPs performed after computing the matrix $M \in \{P_k^-, P_k^+, S_k\}$ of size n_M:

$$\mathcal{H}_M : [M]_{ii} > 0, \quad i = 1, \ldots, n_M.$$

Another issue to be handled by the iIKF is that the interval matrix $[S_k]$ may be non-invertible when at least one real matrix defined using the bounds of the interval matrix is singular. In [Xiong et al. (2013)], an approach based on set inversion using contractors and the algorithm SIVIA (see Section 3) has been suggested to solve this problem. The main idea of this approach is to replace the interval matrix inversion problem in the equation $K_k = P_k^- C_k^T S_k^{-1}$ by constraint propagation problems as follows:

$$[K_k][S_k] = [T_k], \quad (11)$$

where $[T_k] = [P_k^-][C_k]^T$. Every component of the gain matrix $[K_k]$ is considered as an interval in the initial search space defined as:

$$[K_k]_0 = [K_k]_{0_{11}} \times [K_k]_{0_{12}} \times \cdots \times [K_k]_{0_{n_x n_y}}. \quad (12)$$

From (11), a linear equation system with the variables $[K_k]_{ij}$ where $i = 1, \ldots, n_x$, and $j = 1, \ldots, n_y$, is obtained.

The initial search space $[K_k]_0$ can be obtained in several ways. When $[S_k]$ is non-singular, equation $[K_k] = [P_k^-][C_k]^T[S_k]^{-1}$ can still be used as initial search space. In the opposite case, the solution is to use the constraint satisfaction technique starting with a very large initial search space. The contracted subspace is then bisected and tested under SIVIA to eliminate the inconsistent parts. The set of gains given by SIVIA is applied in the correction step to compute the set of state estimates and covariance matrices. The hull of all state estimates and covariance matrices obtained previously are considered as the state estimate and covariance matrix for the current time step.