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Coupling-modulated multi-stability and coherent dynamics in directed networks of heterogeneous nonlinear oscillators with modular topology

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Abstract: We discuss the dynamics of modular networks of coupled nonlinear systems. Networks of this class can be viewed as locally connected clusters or modules of nodes with directed links from one cluster to another. Let these connections form an oriented cycle. Here we show that if connectivity within isolated clusters is diffusive with relatively strong coupling, then spectral properties of the corresponding network coupling matrix promote emergence and co-existence of multiple coherent and orderly dynamic regimes. We hypothesise that, similar to our previous work on the dynamics of cycles, in addition to a nearly fully synchronous state, an attracting rotating wave solution may occur. Furthermore, prevalence of one solution type over the other could be determined by the combination of the cycle length, inter- and intra-cluster coupling strengths, and the number of elements in each module. Our preliminary numerical experiments support these hypotheses.

Keywords: Coupled systems, eigenvalue, synchronization, wave solutions, modular networks, heterogeneous networks, diffusive coupling

1. INTRODUCTION

Understanding the dynamics of interconnected systems of nonlinear ordinary differential equations is arguably amongst the oldest and inspiring problems. Objects of this type occur in a broad range of fields of engineering and science (Strogatz, 2003). Significant progress has been made in this area with regards to general laws governing the emergence of various synchronous states, see e.g. (Pikovsky et al., 2001) and references therein; and the presence of intricate dependencies between network topologies, properties of individual nodes and dynamics in networks have now been elucidated by many authors (Golubitsky et al., 2005), (Pogromsky, 1998), (Scardovi et al., 2010), (Pogromsky et al., 2002), Belykh et al. (2000), (Belykh et al., 2004), (Chandrasekar et al., 2014). Despite this progress, however, a few fundamental questions remain, including the question, how a specific configuration of network topology and weights may affect the overall behavior of the network.

It has been shown recently in Gorban et al. (2015b), Gorban et al. (2015a) that "closing" a chain of identical nonlinear oscillators with directed coupling by adding a directed feedback from the last element in the chain to the first dramatically affects the dynamics of the system. In the chain one only finds a single coherent state, the full synchronization. Closing the chain results in the creation of a new system, in which multiple coherent states may coexist: rotating wave solutions of various modes and a fully synchronous state. The observed abrupt change in dynamics has been attributed to the behavior of the spectrum of the network Laplacian matrix. Furthermore, it has also been shown in Gorban et al. (2015b), Gorban et al. (2015a) that rotating wave solutions prevail in long directed cycles.

In this work we develop and generalize these results in the following two directions. First, instead of directed chains we consider networks with modular structure. Such networks comprise of diffusively and undirectly coupled

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groups of nodes (modules). These groups are linked by directed connections forming a directed cycle. We show that, remarkably, the spectrum of the network Laplacian for such modular structures is closely related to that of individual isolated modules and the corresponding ring or cycle. Similar to our previous work (Gorban et al., 2015b) we hypothesise that rotating wave solutions are likely in such networks. In addition, rotating wave solutions are expected to occur more frequently in the directed cycle of modules than in the directed cycle of simple oscillators. Our preliminary numerical simulations confirm this hypothesis. Second, in addition to nodes with identical dynamics, the result enables us to predict behavior of networks in which individual oscillators differ; their parameters could, for example, be randomly sampled from a distribution centered at fixed nominal values. We hypothesise that, in view of the recent results (Panteley et al., 2015) and provided that coupling within individual modules is strong enough, solutions resembling rotating waves will emerge in this system, too.

The paper is organized as follows. Section 2 contains notational agreements and conventions employed throughout the paper, Section 3 describes the class of systems considered in this work, including network topologies and dynamics of individual nodes. Section 4 contains the main theoretical results, and Section 5 concludes the paper.

2. NOTATION

Throughout the paper the following notational conventions are used

- $\mathbb{N}$ is the set of natural numbers;
- $\mathbb{R}$ denotes the field of real numbers; $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} | \ x \geq 0\}$;
- $\mathbb{R}^n$ stands for the n-dimensional linear space defined over the field of reals;
- $\mathbb{C}$ is the set of complex numbers; let $a \in \mathbb{C}$ then $\Re(a)$ denotes the real part of a, and $\Im(a)$ is the imaginary part;
- let $x \in \mathbb{R}^n$, then $\|x\|$ is the Euclidean norm of x: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$;
- I_N is the $N \times N$ identity matrix;
- Consider a non-autonomous system $\dot{x} = f(x, p, t, u(t))$, where $f : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ is a piecewise continuous function, $p \in \mathbb{R}^d$ is the vector of parameters, $u : \mathbb{R} \to \mathbb{R}^m$ is a continuous function, and $f(\cdot, p, t, u(t))$ is locally Lipschitz; $x(\cdot\ ; t_0, x_0, p, [u])$ stands for the unique maximal solution of the initial value problem: $x(t_0; t_0, x_0, p, [u]) = x_0$. In cases when no confusion arises, we will refer to these solutions as $x(\cdot; t_0, x_0)$, $x(\cdot; x_0)$, or simply $x(\cdot)$. Solutions of the initial value problem above at t are denoted as $x(t; t_0, x_0)$, $x(t; x_0)$, or $x(t)$ respectively.

3. NETWORK DEFINITION AND PRELIMINARIES

3.1 Network structure and isolated node dynamics

In what follows we consider a network of FitzHugh-Nagumo (FHN) (FitzHugh, 1961) oscillators:

$$\begin{cases} \dot{z}_j = \alpha\,(y_j - \beta z_j) \\ \dot{y}_j = y_j - \gamma y_j^3 - z_j + u_j, \end{cases} \tag{1}$$

in which $j = 1, 2, \ldots, n$, is the label of individual node, $n = M \times N$, $M, N \in \mathbb{N}$ will be specified later. For the time being parameters α, β, γ are chosen as

$$\alpha = \tfrac{8}{100},\ \beta = \tfrac{8}{10},\ \gamma = \tfrac{1}{3}.$$

At a later stage we will alow the case in which the values of these parameters are different for each node. Variable u_j corresponds to the network coupling.

The coupling u_j is supposed to satisfy

$$u_j = -\sum_{l=1}^{n} \Gamma_{jl}(\sigma, \mu) y_l, \tag{2}$$

where the matrix $\Gamma(\sigma, \mu) = \{\Gamma_{jl}(\sigma, \mu)\}$ is:

$$\Gamma(\sigma, \mu) = \sigma \Gamma_r(N) \otimes B + I_N \otimes \mu \Gamma_m(M). \tag{3}$$

In (3) matrix B is defined to be

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$\Gamma_m(M)$ is the $M \times M$ symmetric matrix:

$$\Gamma_m(M) = \begin{pmatrix} M-1 & -1 & \cdots & -1 \\ -1 & M-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & M-1 \end{pmatrix} \tag{4}$$

corresponding to the interconnections within each module, and matrix $\Gamma_r(N)$ is the $N \times N$ matrix corresponding to the ring/cycle structure:

$$\Gamma_r(N) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \tag{5}$$

Parameters $\sigma \in \mathbb{R}$ and $\mu \in \mathbb{R}$ are the inter-modular and the intra-modular coupling strengths, respectively. The values of σ, μ are supposed to be constant and non-negative.

A diagram illustrating the connectivity pattern of the class of networks considered is shown in Fig. 1. This network may be viewed as a simple cycle of "supernodes", each corresponding to fully connected modules of identical nodes. Parameters M and N in (4), (5) are hence, respectively the numbers of elements in each modules and the "length" of the cycle formed by the modules. Notice that the coupling matrix $\Gamma(\sigma, \mu)$ is diffusive but not symmetric.

For convenience, let

$$y = (y_1, \ldots, y_n),\ u = (u_1, \ldots, u_n),\ x = (y, z),$$

and $x(\cdot; x_0, \sigma, \mu)$ denote a solution of the coupled system with the coupling strength σ and satisfying the initial condition $x(0) = x_0$.

3.2 Semi-passivity

In addition to the formal definition of node dynamics and network topology, we will rely upon the notions of *semi-passivity* and *strict semi-passivity* (Pogromsky, 1998). For consistency, we recall these notions below