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Robust Network Synchronization of Time-Delayed Coupled Systems

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Abstract: We address the problem of controlled synchronization in networks of time-delayed coupled nonlinear systems. In particular, we prove that, under some mild conditions, there always exists a unimodal region in the parameter space (coupling strength γ versus time-delay τ), such that if γ and τ belong to this region, the systems synchronize. We show how this unimodal region scales with the network topology, which, in turn, provides useful insights of how to design the network topology to maximize robustness against time-delays. The results are illustrated by computer simulations of coupled Hindmarsh-Rose neural chaotic oscillators.

Keywords: Robustness, Delays, Synchronization, Control of Networks, Nonlinear Systems.

1. INTRODUCTION

The emergence of synchronization in networks of coupled dynamical systems is a pervasive phenomenon in various scientific disciplines ranging from biology, physics, and chemistry to social networks and technological applications. Several examples of synchronous behavior in science and engineering can be found in, for instance, Blekhman (1988), Pikovsky et al. (2001), Strogatz (2003), and references therein. One of the first technical results regarding synchronization of coupled nonlinear systems is presented in Fujisaka and Yamada (1983). In this paper, the authors show that coupled chaotic oscillators may synchronize in spite of their high sensitivity to initial conditions. After this result, considerable interest in the notion of synchronization of general nonlinear systems has arisen. Here, we focus on network synchronization of identical nonlinear systems interacting through diffusive time-delayed couplings on networks with general topologies. Time-delayed couplings arise naturally for interconnected systems since the transmission of signals is expected to take some time. Time-delays caused by signal transmission and/or faults in the communication channels affect the behavior of the interconnected systems (e.g., in terms of stability and/or performance).

The results presented here follow the same research line as Steur and Nijmeijer (2010), Steur et al. (2014), Murguia et al. (2015a), and Murguia et al. (2015b), where sufficient conditions for synchronization of diffusively interconnected *semipassive systems* with time-delays are derived. In particular, in Steur and Nijmeijer (2010), the authors prove that, under some mild assumptions, there always exists a region $\mathcal{S}$ in the parameter space (coupling strength γ versus time-delay τ), such that if $(\gamma,\tau) \in \mathcal{S}$ then the systems synchronize. To derive their results, the authors assume that the individual systems are *semipassive* (Pogromsky et al. (1999)) with respect to the coupling variable and the corresponding internal dynamics are *convergent systems*, (see Pavlov et al. (2004)). In the same spirit, here we prove that the region $\mathcal{S}$ is always bounded by a *unimodal function* $\varphi(\gamma)$ defined on some set $\mathcal{J} \subset \mathbb{R}$; and consequently, that there always exists an optimal coupling strength γ^* that leads to the maximum time-delay $\tau^* = \varphi(\gamma^*)$ that can be induced to the network without compromising the synchronous behavior. It follows that for $\gamma = \gamma^*$ and for any $\tau \leq \tau^*$ the systems synchronize, i.e., the gain $\gamma = \gamma^*$ leads to the best tolerance against time-delays of the closed-loop system. Finally, we analyze the effect of the network topology on the values of both the optimal γ^* and the maximum time-delay τ^*, i.e., we show how the eigenvalues of the corresponding Laplacian matrix affect $\varphi(\gamma)$, γ^*, and τ^*. This, in turn, gives insights on how to design the network topology in order to enhance robustness against time-delays.

2. PRELIMINARIES

In this section, we introduce some important properties and definitions which are needed for the subsequent results. Throughout this paper, the following notation is used: the symbol $\mathbb{R}_{>0}(\mathbb{R}_{\geq 0})$ denotes the set of positive (nonnegative) real numbers. The Euclidian norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^T x$. The notation $\text{col}(x_1, ..., x_n)$ stands for the column vector composed of the elements $x_1, ..., x_n$. This notation is also used in case the components x_i are vectors. The $n \times n$ identity matrix is denoted by I_n or simply I if no confusion can arise.

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Likewise, the $n \times m$ matrices composed of only ones and only zeros are denoted as $\mathbf{1}_{n\times m}$ and $\mathbf{0}_{n\times m}$, respectively. The spectrum of a matrix A is denoted by $spec(A)$. The symbol $\otimes$ denotes Kronecker product. Let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$. The space of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. If the functions are (at least) $r \geq 0$ times continuously differentiable, then it is denoted by $\mathcal{C}^r(\mathcal{X}, \mathcal{Y})$. If the derivatives of a function of all orders ($r = \infty$) exist, the function is called smooth and if the derivatives up to a sufficiently high order exist the function is named sufficiently smooth. For simplicity of notation, we often suppress the explicit dependence of time t.

2.1 Communication Graphs

Given a set of interconnected systems, the communication topology is encoded through a communication graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ denote a weighted undirected graph, where $\mathcal{V} = \{v_1, v_2, ..., v_k\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A \in \mathbb{R}^{k\times k} := a_{ij}$ is the weighted *adjacency matrix* with entries $a_{ij} = a_{ji} \geq 0$. The set of neighbors of v_i, denoted as $\mathcal{E}_i$, is the set of nodes which have edges pointing to v_i. Throughout this manuscript, it is assumed that the communication graph is *strongly connected* and *simple*, see Bollobas (1998). We denote the *degree matrix* $D \in \mathbb{R}^{k\times k} := diag\{d_1, ..., d_k\}$ with $d_i := \sum_{j\in\mathcal{E}_i} a_{ij}$, and $L := D - A$, which is called the *Laplacian matrix* of the graph $\mathcal{G}$.

2.2 Semipassive Systems

Consider the system

$$\dot{x} = f(x, u), \tag{1a}$$

$$y = h(x), \tag{1b}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$, and sufficiently smooth functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 1. Pogromsky et al. (1999). System (1) is called $\mathcal{C}^r$*-semipassive* if there exists a nonnegative storage function $V \in \mathcal{C}^r(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ such that $\dot{V} \leq y^T u - H(x)$, where the function $H \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ is nonnegative outside some ball, i.e., $\exists\, \varphi > 0$ s.t. $|x| \geq \varphi \to H(x) \geq \varrho(|x|)$, for some continuous nonnegative function $\varrho(\cdot)$ defined for $|x| \geq \varphi$. If the function $H(\cdot)$ is positive outside some ball, then the system (1) is said to be *strictly* $\mathcal{C}^r$*-semipassive*.

A (strictly) $\mathcal{C}^r$-semipassive system behaves like a (strictly) passive system for large $|x(t)|$.

2.3 Convergent Systems

Consider system (1a) and suppose $f(\cdot)$ is locally Lipschitz in x, $u(\cdot)$ is piecewise continuous in t and takes values in some compact set $u \in U \subseteq \mathbb{R}^m$.

Definition 2. System (1a) is said to be *convergent* if and only if for any bounded signal $u(t)$, defined on the whole interval $(-\infty, +\infty)$, there is a unique bounded globally asymptotically stable solution $\bar{x}_u(t)$, defined in the same interval, for which it holds that $\lim_{t\to\infty} |x(t) - \bar{x}_u(t)| = 0$ for all initial conditions.

For a *convergent system*, the limit solution is solely determined by the external excitation $u(t)$ and not by the initial condition. A sufficient condition for system (1a) to be convergent is presented in the following proposition.

Proposition 1. Demidovich (1967) and Pavlov et al. (2004). If there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that all the eigenvalues $\lambda_i(Q)$ of the symmetric matrix:

$$Q(x, u) = \frac{1}{2}\left(P\left(\frac{\partial f}{\partial x}(x, u)\right) + \left(\frac{\partial f}{\partial x}(x, u)\right)^T P\right), \tag{2}$$

are negative and separated from zero, i.e., there exists a constant $c \in \mathbb{R}_{>0}$ such that $\lambda_i(Q) \leq -c < 0$, for all $i \in \{1, ..., n\}$, $u \in U$, and $x \in \mathbb{R}^n$, then system (1a) is globally exponentially convergent. Moreover, for any pair of solutions $x_1(t), x_2(t) \in \mathbb{R}^n$ of (1a), it is satisfied that: $\frac{d}{dt}\left((x_1 - x_2)^T P(x_1 - x_2)\right) \leq -\alpha\, |x_1 - x_2|^2$ with constant $\alpha := \frac{c}{\lambda_{\max}(P)}$ and $\lambda_{\max}(P)$ being the largest eigenvalue of the symmetric matrix P.

3. PROBLEM STATEMENT

Consider k identical nonlinear systems of the form

$$\dot{\zeta}_i = q(\zeta_i, y_i), \tag{3}$$

$$\dot{y}_i = a(\zeta_i, y_i) + CBu_i, \tag{4}$$

with $i \in \mathcal{I} := \{1, ..., k\}$, state $x_i := \mathrm{col}(\zeta_i, y_i) \in \mathbb{R}^n$, internal state $\zeta_i \in \mathbb{R}^{n-m}$, output $y_i \in \mathbb{R}^m$, input $u_i \in \mathbb{R}^m$, sufficiently smooth functions $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$ and $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$, matrices $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$, and the matrix $CB \in \mathbb{R}^{m\times m}$ being similar to a positive definite matrix. For the sake of simplicity, it is assumed that $CB = I_m$ (results for the general case with CB being similar to a positive definite matrix can be easily derived). The system (3),(4) is assumed to be *strictly* $\mathcal{C}^1$*-semipassive* and the internal dynamics (3) is assumed to be an *exponentially convergent system.* Let the k systems (3),(4) interact on simple strongly connected graph through the *diffusive time-delayed coupling*

$$u_i(t) = \gamma \sum_{j\in\mathcal{E}_i} a_{ij}\left(y_j(t - \tau) - y_i(t - \tau)\right), \tag{5}$$

where $\tau \in \mathbb{R}_{\geq 0}$ denotes a constant time-delay, $y_j(t - \tau)$ and $y_i(t - \tau)$ are the delayed outputs of the j-th and i-th systems, $\gamma \in \mathbb{R}_{\geq 0}$ denotes the coupling strength, $a_{ij} \geq 0$ are the weights of the interconnections, and $\mathcal{E}_i$ is the set of neighbors of system i. It is assumed that the graph is *undirected*, i.e., $a_{ij} = a_{ji}$. Since the coupling strength is encompassed in the constant γ, it is assumed without loss of generality that $\max_{i\in\mathcal{I}} \sum_{j\in\mathcal{E}_i} a_{ij} = 1$. Note that all signals in coupling (5) are time-delayed. Such a coupling may arise, for instance, when the systems are interconnected through a centralized control law. Coupling (5) can be written in matrix form as $u = -\gamma\,(L \otimes I_m)\, y(t - \tau)$ with $u := \mathrm{col}(u_1, \ldots, u_k)$, $y := \mathrm{col}(y_1, \ldots, y_k)$, and Laplacian matrix $L = L^T \in \mathbb{R}^{k\times k}$. The authors in Steur and Nijmeijer (2010) prove that the k coupled systems (3)-(5) asymptotically synchronize provided that γ is sufficiently large and the product of the coupling strength and the time-delay $\gamma\tau$ is sufficiently small. It follows that there exists a region $\mathcal{S}$ in the parameter space, such that if $(\gamma, \tau) \in \mathcal{S}$, the systems synchronize. In this manuscript, we go one step further by showing that this region $\mathcal{S}$ is actually bounded by a *unimodal function* $\varphi : \mathcal{J} \to \mathbb{R}_{\geq 0}$, $\gamma \mapsto \varphi(\gamma)$. Hence, there exists an optimal coupling