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Data rate limitations for observability of nonlinear systems ⋆

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Abstract: The paper deals with observation of nonlinear and deterministic, though maybe chaotic, discrete-time systems via finite capacity communication channels. We consider various types of observability, and offer new tractable analytical techniques for both upper and lower estimation of the threshold that separates data rates for which reliable state observation is and is not possible, respectively. The main results are illustrated via their application to two celebrated samples of chaotic systems associated with the logistic and Lozi maps, respectively. In these cases, the thresholds attributed to some of the considered notions of observability are found in a closed form; they are shown to continuously depend on the parameters of the Lozi system.

Keywords: Entropy, Nonlinear systems, Lyapunov function

1. INTRODUCTION

Recent advances in communication technology have created the possibility of large-scale control systems, where the control tasks are distributed over many agents orchestrated via a communication network; a particular example can be found in modern industrial systems, where the components are often connected over digital band-limited serial communication channels. This motivated development of a new chapter of control theory, where control and communication issues are integrated; see e.g., (Murray (2002); Antsaklis and Baillieul (2007); Matveev and Savkin (2009); Mahmoud (2014); Yüksel and Basar (2013); Postoyan and Nešić (2012); Postoyan et al. (2014)).

In this area, one of the key questions is about the smallest communication data rate required to achieve a certain control objective for a given plant. This fundamental parameter has been studied in a variety of settings (Baillieul (2004); de Persis and Isidori (2004); Nair et al. (2004); Liberzon and Hespanha (2005); de Persis (2005); Savkin (2006); Matveev and Savkin (2009); Nair et al. (2007)) and always found to be somewhat similar to the topological entropy, which is an ubiquitous quantitative measure of randomness, chaos, uncertainty, and complexity in dynamical systems (Katok (2007); Donarowicz (2011)). These studies gave rise to specialized control-oriented concepts of entropy Nair et al. (2004); Savkin (2006); Colonius and Kawan (2009); Hagihara and Nair (2013); Colonius and Kawan (2011); Kawan (2011); Colonius et al. (2013). Most close to the canonical definitions (Adler et al. (1965); Bowen (1971); Dinaburg (1970)) of the topological entropy is the concept accounting for uncertainties in the plant model (Savkin (2006)).

For nonlinear systems, studies of topological entropy of smooth dynamical systems are often much concerned with the upper Lyapunov exponents of the associated linear time-varying systems. In general, topological entropy shares a common unpleasant feature with these exponents, i.e., possible discontinuity with respect to the system's parameters. For example, Buzzi (2009) showed that for piecewise affine surface homeomorphisms, the topological entropy is lower semicontinuous, while Yildiz (2012) found a line segment in the parameter space of the Lozi map over which the topological entropy is not upper semicontinuous. This discontinuity is a serious complication for control oriented problems, where a certain degree of robustness is required as a must for almost all applications.

This paper provides a first evidence that an approach to resolution of this problem can be acquired via a better insight into the concerned performance issues. To this end, we consider the state estimation problem for generic smooth deterministic though maybe, chaotic, nonlinear systems. This is of interest in its own right and since many control problems can be typically solved if a reliable state estimate is available. We also introduce and study three notions of observability via a finite capacity communication channel. Observability in the slightest sense (mere observability) means that the state can be estimated with as high accuracy as desired, but does not exclude that the initial accuracy may drastically degrade over time. Observability in a stronger sense means that the accuracy can be kept proportional to its initial value over the infinite time horizon. The third notion is somewhat similar to the uniform asymptotic observability and additionally requires that the estimation error exponentially decays to zero as time progresses. Each of these concepts is

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associated with a threshold that separates channel data rates for which reliable state observation in the concerned sense is and is not possible, respectively. We show that for mere observability, this threshold is given in terms of the topological entropy of the observed system and does not exceed two other thresholds, which are typically equal to each other. The paper offers tractable analytical techniques for both upper and lower estimation of these thresholds. They are in the vein of the second Lyapunov method, unlike the bulk of the literature on estimation of the topological entropy and the likes.

The benefits from these techniques are illustrated for two celebrated samples of discrete-time dynamical systems, i.e., the logistic and Lozi systems. Despite their simplicity, they exhibit a remarkably complex behavior. In these cases, the above techniques permit us to explicitly compute the entropy-like data-rate thresholds for the two stronger notions of observability in a closed form. It appears that for the Lozi map, these thresholds depend on the system's parameters continuously. This overcomes the "discontinuity" trouble inherent in topological entropy, creates a prerequisite for robust communication schemes and gives some promise for possible extension of this "continuity" phenomenon on other systems. Anyhow, the proposed upper estimates related to two stronger notions of observability are featured with such continuity in general, basically because of their roots in the second Lyapunov method.

The paper is organized as follows. In Sects. 2 and 3, we pose the problem and present preliminary results and basic definitions. Sect. 4 deals with constructive data rate estimates that guarantee solvability of the observation problem in various contexts. Sect. 5 illustrates these results via their application to the logistic and Lozi systems, which in fact, constitutes the major contribution of the paper. Sect. 7 offers brief conclusions.

2. STATE ESTIMATION PROBLEM AND BASIC DEFINITIONS

We consider a discrete-time invariant nonlinear system

$$x(t+1) = \phi[x(t)] \quad t \in \mathbb{Z}_+, \qquad x(0) \in K, \tag{1}$$

where $x(t)$ is the state and $\mathbb{Z}_+$ is the set of nonnegative integers, whereas $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ are a given continuous map and nonempty compact set of initial states, respectively. Let ϕ^k stand for the kth iterate of the map ϕ, and

$$K(t) := \phi^t(K), \quad K^\infty := \bigcup_{t \geq 0} K(t). \tag{2}$$

It is needed to build an accurate estimate $\widehat{x}(t)$ of $x(t)$ at a remote location, where direct observation of $x(t)$ is impossible.

The only way to deliver data from the sensor to this location is via a discrete communication channel. At time t, it carries a discrete-valued symbol $e(t)$. So to be transmitted, continuous-valued sensor readings $x(t)$ should be first converted into such symbols by a special device, called the *coder*. Its outputs are communicated for the unit time across the channel to a *decoder* that produces an estimate $\widehat{x}(t) \in \mathbb{R}^n$ of the current state $x(t)$; see Fig. 1. Thus the *observer* consists of the coder and decoder, which are described by the following respective equations:

$$\begin{aligned} e(t) &= \mathcal{C}[t, x(0), \dots, x(t) | \widehat{x}(0), \delta], \quad t \geq 0, \\ \widehat{x}(t) &= \mathcal{D}\left[t, e(0), \dots, e(t-1) | \widehat{x}(0), \delta\right], \quad t \geq 1. \end{aligned} \tag{3}$$

Here $\widehat{x}(0)$ is an initial estimate, and $\delta > 0$ is its accuracy:

$$\|x(0) - \widehat{x}(0)\| < \delta. \tag{4}$$

We assume that both coder and decoder are initially given common values of $\widehat{x}(0)$ and δ, and are aware of $\phi(\cdot)$ and K.

We are mainly interested in the case where solutions of the observed system (1) are unstable: without utilization of data transmitted via the communication channel, even a very small initial discrepancy between the state and its estimate may critically increase as time progresses. This for example, holds if K is an invariant chaotic set, which situation is our major interest. So, we attempt to find an answer to the following question. *How much data should be communicated per unit time to avoid a detrimental degrade of the estimation accuracy?*

In doing so, we address an averaged amount of data. So we borrow the model of communication channel from (Matveev and Savkin (2005)), which allows for transmission delays, dropouts, and non-stationary instant data rate, (see (Matveev and Savkin, 2009, Sect. 3.4) for details). Specifically, we assume that within any time interval of duration r, no less than $b_-(r)$ but no more than $b_+(r)$ bits of data can be transmitted across the channel, and also that the respective per unit time rates converge to a common value c called the *channel capacity*:

$$r^{-1}b_-(r) \to c \quad \text{and} \quad r^{-1}b_+(r) \to c \quad \text{as} \quad r \to \infty. \tag{5}$$

Following (Matveev and Pogromsky (2016)), we also introduce three notions of successful observation.

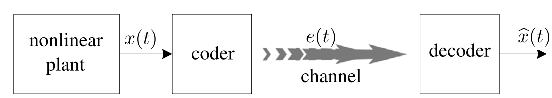

Fig. 1. State estimation over a finite capacity channel.

Definition 1. For the observer (3), $\varepsilon > 0$ is called the anytime exactness of observation of the system (1) if there exists $\delta(\varepsilon, K) > 0$ such that

$$\|x(t) - \widehat{x}(t)\| \leq \varepsilon \qquad \forall t \in \mathbb{Z}_+ \tag{6}$$

whenever $x(0), \widehat{x}(0) \in K$ and the initial error is small enough: (4) holds with $\delta = \delta(\varepsilon, K) > 0$.

However, this is consistent with severe degradation of accuracy: $\varepsilon \gg \delta$. Now we introduce a class of observers for which such degradation does not hold: the anytime exactness is at least proportional to the initial accuracy.

Definition 2. The observer (3) is said to *regularly observe the system* (1) if there exist $\delta_* > 0$ and $G > 0$ such that for all $\delta < \delta_*$, all $x(0), \widehat{x}(0) \in K$ satisfying (4), and all $t \geq 0$, the estimation error obeys the inequality $\|x(t) - \widehat{x}(t)\| \leq G\delta$.

The observers from the next class not only avoid drastic regress of accuracy but also restore and improve the initial accuracy as time progresses.

Definition 3. The observer is said to *finely observe the system* (1) if the observation error is uniformly proportional to the initial error and also exponentially decays to zero as time progresses: there exist $\delta_* > 0$, $G > 0$, and $g \in (0,1)$ such that for all $\delta < \delta_*$ and all $x(0), \widehat{x}(0) \in K$ satisfying (4), the following inequality holds

$$\|x(t) - \widehat{x}(t)\| \leq G\delta g^t \qquad \forall t \geq 0. \tag{7}$$

Definition 4. The system (1) is said to be *i) observable, ii) regularly, and iii) finely observable via a communication channel* if the following respective claims hold (where the observers are meant to operate via the channel at hands):

i) For any $\varepsilon > 0$, there exists an observer (3) that observes the system (1) with anytime exactness ε;