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Synchronization in Ring Networks of Systems with Transmission Delays ⋆

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Abstract: This paper investigates synchronization phenomena that occur in ring networks of identical chaotic systems with delays. In particular, we focus on delay-independent synchronization of systems in ring networks with transmission delays. First, we show that ring networks consisting of an odd number of systems emerge full synchronization for coupling strengths larger than a certain value independently on the size of time-delay. Next, we also show that if the number of systems in ring networks is even, then partial synchronization can be observed regardless of the size of time-delay. In this case, full synchronization may also be observed as a special case of partial synchronization. Numerical examples show the validity of these theoretical results.

Keywords: Synchronization, Time-delay, Network systems, Complex systems, Nonlinear systems

1. INTRODUCTION

Synchronization has been widely investigated in various fields including applied physics, mathematical biology, control engineering and science, and social sciences (Pikovsky et al. (2003); Strogatz (2003)). Chaos synchronization was discovered by Fujisaka and Yamada (Fujisaka and Yamada (1983)) and came to public attention by Pecora and Carrol (Pecora and Carroll (1990)). Since then, synchronization has been receiving increasing attention, and the attention is also spreading from the master-slave type synchronization to network synchronization, synchronization between hetero systems and so on. Synchronization of coupled systems with delays is also one of the widely studied problems (Amano et al. (2004); Oguchi et al. (2008); Steur et al. (2012)). In general, the synchronization conditions depends on the size of time-delays and the coupling strengths between systems, but depending on the coupling type and network structures, the condition may be independent of the size of time-delay. This implies that under some conditions, the coupled systems can synchronize regardless of any length of time-delays in the couplings without any delay compensation.

This paper investigates delay-independent synchronization, which occurs regardless of the size of time-delay, in networks of systems with transmission delays. The possibility of such synchronization in networks of systems with transmission delays was pointed out in several works (c.g. Lu et al. (2006)), but the relationship between the network structure and delay-independent synchronization is not clear. Therefore, we focus on network systems in which the structure is ring topology and each coupling has a transmission delay. For such systems, we show that if the number of systems is odd, then full synchronization does not depend on time-delay for a coupling strength larger than a certain value. Also, we show that if the number of systems is even, then delay-independent partial synchronization appears. Of course, since the synchronization condition is obtained by analyzing the stability of the synchronization error dynamics, if the synchronization error dynamics is given by differential equations, the synchronization condition is independent of time-delay (Yanagi and Oguchi (2014)). However, ring network systems to be considered in this paper are not such a class of network systems.

2. RING NETWORKS OF SYSTEMS WITH TRANSMISSION DELAYS

Throughout this paper, we consider the following N identical nonlinear systems.

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \tag{1}$$

where $i \in \mathcal{I} = \{1, \dots, N\}$, and $x_i \in \mathbb{R}^n$, u_i and $y_i \in \mathbb{R}$ are the state, the input and the output, respectively. $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth, and B and C are constant matrices with suitable dimensions, and without loss of generality, it is assumed that CB is strictly positive.

Now we introduce the definitions of full synchronization and partial synchronization in network systems, respectively.

Definition 1. (Oguchi and Nijmeijer (2011)) *If there exists a positive real number r such that the trajectories $x_i(t)$ of the systems (1) with the initial conditions φ_i, φ_j such that*

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$\|\varphi_i - \varphi_j\|_C \le r$ satisfy $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$ for all i, j, then all systems in the network are asymptotically synchronized. Here $\|\varphi\|_c := \max_{-\tau \le \theta \le 0} \|\varphi(\theta)\|$ stands for the norm of a vector function φ, where $\|\cdot\|$ refers to the Euclidean vector norm.

Definition 2. (Mimura and Oguchi (2012)) *If there exists a positive real number r such that the trajectories $x_i(t)$ and $x_j(t)$ of a part of subsystems i and j in networks with the initial conditions φ_i, φ_j such that $\|\varphi_i - \varphi_j\|_C \le r$ satisfy $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$ for i, j, then the subsystems i and j are asymptotically synchronized and the network systems is called to be partially synchronized.*

From the assumption that CB is strictly positive, which means that the system has relative degree one, this system is rewritten in the following normal form by applying a global coordinate transformation(Pogromsky and Nijmeijer (2001)).

$$\begin{cases} \dot{z}_i(t) = q(z_i(t), y_i(t)) & \text{(2a)} \\ \dot{y}_i(t) = a(y_i(t), z_i(t)) + bu_i(t) & \text{(2b)} \end{cases}$$

where $z_i \in \mathbb{R}^{n-1}$, $b = CB$, $q : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ and $a : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$.

Throughout this paper, we assume that the following assumptions hold for the system (2):

Assumption 1. (Pogromsky and Nijmeijer (2001)) Each system (2) for $i = 1, \dots, N$ is strictly $\mathcal{C}^1$-semi-passive, i.e., there exists a radially unbounded positive definite storage function $V \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\dot{V}(x_i(t)) \le y_i(t)u_i(t) - H(x_i(t))$$

where $H : \mathbb{R}^n \to \mathbb{R}$ is a scalar function positive outside some ball $\mathcal{B} = \{x_i \in \mathbb{R}^n \mid \|x_i\| < R\}$.

Assumption 2. (Pogromsky and Nijmeijer (2001)) For the sub-dynamics (2a) in each system (2), there exists a positive definite function $V_0 \in \mathcal{C}^2(\mathbb{R}^{n-1}, \mathbb{R}_+)$ such that

$$(\nabla V_0(\bar{z}_{ij}))^\top (q(z_i, y^*) - q(z_j, y^*)) \le -\alpha \|\bar{z}_{ij}\|^2$$

for any $z_i, z_j \in \mathbb{R}^{n-1}$, $y^* \in \mathbb{R}$, where $\bar{z}_{ij} = z_i - z_j$ and $\alpha > 0$.

Under these assumptions, we consider networks of systems (2) coupled by the following inputs.

$$u_i(t) = \sum_{j=1, j \ne i}^{N} k_{ij}(y_j(t - \tau) - y_i(t)) \tag{3}$$

where τ is a constant time-delay, and k_{ij} denotes the coupling strength from system j to system j such that if there exist a coupling from system j to system i, $k_{ij} = k \ne 0$, and otherwise $k_{ij} = 0$.

The system (2) for $i = 1, \dots, N$ are rewritten as

$$\begin{cases} \dot{z}(t) = q(z(t), y(t)) \\ \dot{y}(t) = a(y(t), z(t)) - kbu(t) \end{cases} \tag{4}$$

where $z = col(z_1, \dots, z_N) \in \mathbb{R}^{N(n-1)}$, $y = (y_1, \dots, y_N)^\top \in \mathbb{R}^N$, $u = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, and vector fields q and a are given by

$$q(z, y) = col(q(z_1, y_1), \cdots q(z_N, y_N)),$$
$$a(y, z) = col(a(y_1, z_1), \cdots, a(y_N, z_N)).$$

Substituting $u(t) = -k(Dy(t) - Ay(t - \tau))$ into the system (4), the total dynamics of system (2) coupled by (3) is described by

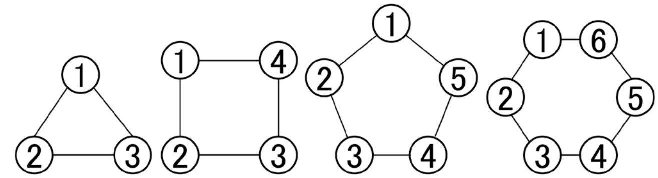

Fig. 1. Ring networks ($N = 3, 4, 5, 6$)

$$\begin{cases} \dot{z}(t) = q(z(t), y(t)) \\ \dot{y}(t) = a(y(t), z(t)) - kb(Dy(t) - Ay(t - \tau)), \end{cases} \tag{5}$$

where matrices $D \in \mathbb{R}^{N \times N}$ and $A \in \mathbb{R}^{N \times N}$ are the degree and adjacency matrices of graph $\mathcal{G}$, respectively.

Thanks to Assumption 1, it is guaranteed that all solutions of systems (1) coupled by (3), i.e. the solution of system (5), are ultimately bounded independently of the coupling strength k and time-delay τ (Steur and Nijmeijer (2011)).

For this coupled system, we discuss the relationship between the network topology and delay-independent synchronization in the following sections.

3. DELAY INDEPENDENT SYNCHRONIZATION AND RING NETWORKS

This section considers synchronization in network systems introduced in the preceding section. In particular, we investigate synchronization whose condition is independent of the values of time-delays in couplings. Yanagi and Oguchi (2014) clarified the network structures such that the synchronization error dynamics between two systems in the network is described by a differential equation without delay. The derived condition on network structure is a sufficient condition for synchronization to be independent of delay, but it is not necessary.

In this paper, we show that delay-independent synchronization occurs even if the delay error dynamics is represented by delay differential equations. To show the fact, we focus on only bidirectional ring network systems as shown in Fig. 1.

3.1 Ring network with an odd number of nodes

First, we consider ring networks with an odd number of nodes, i.e. $N = 2M + 1$ where $M \in \mathbb{N}$. Then, all nodes in bidirectional ring networks have degree two, that is $\sum_{j=1, j \ne i}^{N} a_{ij} = 2$ for any i, where a_{ij} denotes the (i, j)-entry of the adjacency matrix A, and the degree matrix D is given by $D = diag(2, \dots, 2)$. Then

$$\mathcal{M} = \{col(z, y) \in \mathbb{R}^{Nn} | z_i = z_j \text{ and } y_i = y_j, \forall i, j \in \mathcal{I}\}$$

is a linear invariant manifold for system (5), and $\mathcal{M}$ is called the synchronization manifold. This means that if $x(t_0 + \theta) \in \mathcal{M}$ for $\theta \in [-\tau, 0]$, $x(t) \in \mathcal{M}$ for $t \ge t_0$.

Concerning full synchronization in ring networks with an odd number of systems, we obtain the following result.

Theorem 3. Consider bidirectional ring networks consisting of an odd number of systems (2) with delayed couplings (3). Under Assumption 1 and Assumption 2, there exists a positive constant $\bar{k}$ such that if $k > \bar{k}$, then all systems