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Boundary Energy Control of the Sine-Gordon Equation ⋆

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Abstract: The boundary energy control problem for the sine-Gordon equation is posed. Two control laws solving this problem based on Speed-Gradient method with smooth and nonsmooth goal functions are proposed. The control law obtained via a nonsmooth goal function is proved to steer the system to any required nonzero energy level in finite time. The results of the numerical evaluation of the proposed algorithms demonstrate attainability of the control goal both for the cases of decreasing and increasing the system energy and show high rate of vanishing of the control error.

Keywords: boundary control, energy control, Speed-Gradient, sine-Gordon equation.

1. INTRODUCTION

The problems of energy control form an important class of control problems for physical systems. For nonlinear controlled systems their systematic study started in Fradkov (1996); Fradkov et al. (1997); Fradkov and Pogromsky (1998) where finite-dimensional Hamiltonian systems were studied and efficiency of the speed-gradient method for their solution was demonstrated. From the first glance, the energy control goal is soft compared to regulation or tracking problems, since it means stabilization of the constant energy hypersurface compared to stabilization of 0-dimensional or 1-dimensional goal manifold, correspondingly. However, the problem is often complicated due to complicated geometry of the constant energy surface that does not permit global reduction to a linear formulation. The problem has numerous applications: stabilization of unstable pendulums Astrom and Furuta (2000), control of vibration set ups Blekhman et al. (2001), escape from a potential well Fradkov (1999), controlled dissociation of molecules Fradkov (2007), etc. However, in most studied cases the controlled system model is finite-dimensional.

First time, infinite-dimensional energy control problems were considered in the book Fradkov (2007) where the speed-gradient energy control algorithm for the sine-Gordon equation was proposed, and Aero et al. (2006) where control of optical mode for a complex crystalline lattice was studied. However first rigorous results were obtained only a decade later Andrievsky et al. (2016). In Porubov et al. (2015) the problem of controlling travelling waves for sine-Gordon equation was investigated and some robustness of the system controlled by Speed-Gradient algorithm with respect to initial conditions was established.

However, the existing studies deal with distributed control action. Boundary energy control was not studied before. Boundary control is an important problem both from practical point of view, and as a challenging theoretical problem Butkovskii (1969). Boundary control of the sine-Gordon system was studied in detail for the regulation and tracking problems by Kobayashi (see Kobayashi (2003a,b, 2004)). However, energy control problem was not dealt with before.

This paper is devoted to the boundary Speed-Gradient energy control of the sine-Gordon equation. Analytical conditions for achievement of the control goal are established, and numerical experiments illustrating transient properties of the closed loop system are presented.

The paper is organized as follows. A formal statement of the boundary energy control problem for the one-dimensional sine-Gordon is given in Section 2. In Section 3, a required control law is designed with the use of the smooth Speed-Gradient algorithm, and its asymptotic convergence is proved. In Section 4, we utilize the nonsmooth Speed-Gradient algorithm in order to design a control law that steers the system to any prespecified nonzero energy level in finite time. Finally, results of numerical experiments are presented in Section 5.

2. PROBLEM FORMULATION

Consider the one-dimensional sine-Gordon equation with the following initial and boundary conditions

$$z_{tt}(t,x) - kz_{xx}(t,x) + \beta \sin z(t,x) = 0, \quad (t,x) \in \Omega, \quad (1)$$

$$z(0,x) = z^0(x), \quad z_t(0,x) = z^1(x), \quad x \in [0,1], \quad (2)$$

$$z(t,0) = 0, \quad z_x(t,1) = u(t), \quad t \geq 0, \quad (3)$$

$$y(t) = \big(z_t(t,1), H(z(t))\big) \quad t \geq 0,$$

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where $\beta \geq 0$ and $k > 0$ are given parameters, $\Omega = [0, +\infty) \times [0, 1]$, $z^0, z^1 \colon [0, 1] \to \mathbb{R}$ are given functions, $u(t)$ is a control input, $y(t)$ is the output, and

$$H(z) = \frac{1}{2} \int_0^1 \Big(z_t^2 + k z_x^2 + 2\beta \big(1 - \cos z\big) \Big) \, dx$$

is the Hamiltonian for the equation (1).

We pose the control problem as finding the control law $u(t)$, which ensures the control objective

$$H(z(t)) \to H^*, \tag{4}$$

where $z(t)$ is a solution of (1)–(3), and $H^* \geq 0$ is prespecified. Thus, the control objective is to reach the desired energy level H^* in the system governed by the sine-Gordon equation (1)–(3).

It should be noted that all results below can be easily modified to the case

$$z_t(t, 1) = u(t), \quad y(t) = \big(z_x(t, 1), H(z(t))\big) \quad t \geq 0,$$

i.e. one can "swap" $z_t(t, 1)$ and $z_x(t, 1)$.

3. SMOOTH SPEED-GRADIENT ALGORITHM

In this section, we design a required control law with the use of the Speed-Gradient algorithm in finite form (see, e.g., Fradkov et al. (1999); Fradkov (2007)).

Introduce the following goal function

$$Q(z(t)) = \frac{1}{2} (H(z(t)) - H^*)^2, \quad t \geq 0.$$

The derivative of this function along solutions of the system (1)–(3) has the form

$$\begin{aligned}
&\frac{d}{dt} Q(z(t)) \\
&= (H(z(t)) - H^*) \int_0^1 \Big(z_t z_{tt} + k z_x z_{xt} + \beta \sin z z_t \Big) \, dx \\
&\qquad = (H(z(t)) - H^*) \\
&\times \Bigg[\int_0^1 \big(z_t (k z_{xx} - \beta \sin z) - k z_{xx} z_t + \beta \sin z z_t \big) \, dx \\
&\qquad + k z_x(t, 1) z_t(t, 1) - k z_x(t, 0) z_t(t, 0) \Bigg] \\
&\qquad = (H(z(t)) - H^*) k u(t) z_t(t, 1).
\end{aligned}$$

Then according to the Speed-Gradient algorithm, one defines the control law $u(t)$ as follows

$$u(t) = -\gamma \frac{\partial}{\partial u} \frac{dQ(z)}{dt} = -\gamma (H(z(t)) - H^*) k z_t(t, 1),$$

where $\gamma > 0$ is a scalar gain. However, we will consider the more general control algorithm of the form

$$u(t) = -\gamma \psi(H(z(t)) - H^*) z_t(t, 1), \tag{5}$$

where $\psi \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\psi(0) = 0$ and $\psi(s)s > 0$ for any $s \in \mathbb{R}$ (cf. Shiriaev et al. (2001)). Note that for any solution z of the closed-loop system (1)–(3), (5) one has

$$\frac{d}{dt} H(z(t)) = -\gamma \psi(H(z(t)) - H^*) k z_t^2(t, 1),$$

which yields

$$\frac{d}{dt} H(z(t)) \begin{cases} \leq 0, & \text{if} \quad H(z(t)) > H^*, \\ = 0, & \text{if} \quad H(z(t)) = H^*, \\ \geq 0, & \text{if} \quad H(z(t)) < H^* \end{cases} \tag{6}$$

Therefore $H(z(t)) \in \operatorname{co}\{H^*, H(z(0))\}$ for all $t \geq 0$.

Hereinafter, we suppose that for any sufficiently smooth initial data z^0 and z^1 there exists a sufficiently smooth solution of the problem (1)–(3), (5). The proof of the fact that such solution exists is outside the scope of the paper, and will be published elsewhere.

Let us study the performance of the control system (1)–(3) with the proposed control algorithm (5).

Theorem 1. For any sufficiently smooth initial data z^0, z^1 such that $H(z(0)) \neq 0$, and for all $k > 0$, $0 \leq \beta < k\pi^2/4$ and $\gamma > 0$ one has

$$H(z(t)) \to H^* \quad \text{as} \quad t \to \infty,$$

where z is a solution of the system (1)–(3), (5).

Proof. (Sketch.) For any $\varepsilon > 0$ introduce the Lyapunov-like function

$$V(t) = H(z(t)) + \varepsilon \operatorname{sign}(H(z(t)) - H^*) g(t),$$

where $\operatorname{sign}(0) = 0$ and

$$g(t) = \int_0^1 x z_t z_x \, dx.$$

Applying the inequalities

$$\begin{aligned}
&\Big| \int_0^1 x z_t z_x \, dx \Big| \leq \int_0^1 |z_t z_x| \, dx \\
&\quad \leq \frac{1}{2} \int_0^1 z_t^2 \, dx + \frac{1}{2} \int_0^1 z_x^2 \, dx \leq \max \left\{ 1, \frac{1}{k} \right\} H(z(t)),
\end{aligned}$$

one obtains that

$$0 \leq (1 - \varepsilon k_0) H(z(t)) \leq V(t) \leq (1 + \varepsilon k_0) H(z(t)) \tag{7}$$

for all $t \geq 0$ and $\varepsilon \in (0, 1/k_0)$, where $k_0 = \max\{1, 1/k\}$.

Arguing in the same way as in Kobayashi (2003a, 2004) one can show that for any $0 \leq \beta < k\pi^2/4$ there exists $C_0 > 0$ such that

$$\frac{d}{dt} g(t) \leq -C_0 H(t) + \frac{1}{2} z_t^2(t, 1) + \frac{k}{2} z_x^2(t, 1), \tag{8}$$

Note also that for any $t \geq 0$ such that $H(z(t)) \neq H^*$ one has

$$\begin{aligned}
\frac{d}{dt} V(t) = &-\gamma \psi(H(z(t)) - H^*) k z_t^2(t, 1) \\
&+ \varepsilon \operatorname{sign}(H(t) - H^*) \frac{d}{dt} g(t).
\end{aligned} \tag{9}$$

Let $\Delta > 0$ be arbitrary, and suppose that $H(z(0)) > H^*$. Clearly, there exists $T_\Delta \in [0, +\infty]$ such that $H(z(t)) \geq H^* + \Delta$ for any $t \in [0, T_\Delta)$, and $H(z(t)) < H^* + \Delta$ for any $t \in (T_\Delta, +\infty)$ (see (6)). Our aim is to show that $T_\Delta < +\infty$ for any $\Delta > 0$.

Arguing by reductio ad absurdum, suppose that there exists $\Delta > 0$ such that $T_\Delta = +\infty$. Denote

$$\psi_\Delta := \min \Big\{ \psi(s) \;\Big|\; s \in \big[\Delta, H(z(0)) - H^*\big] \Big\} > 0$$

and

$$\Psi_\Delta := \max \Big\{ \psi(s) \;\Big|\; s \in \big[\Delta, H(z(0)) - H^*\big] \Big\} > 0$$

Then for any $t \geq 0$ one has $\Psi_\Delta \geq \psi(H(z(t)) - H^*) \geq \psi_\Delta$. Therefore taking into account (9), (8) and (7) one obtains that