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Tutorial Article

Max-plus algebra in the history of discrete event systems☆

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ABSTRACT

This paper is a survey of the history of max-plus algebra and its role in the field of discrete event systems during the last three decades. It is based on the perspective of the authors but it covers a large variety of topics, where max-plus algebra plays a key role.

Contents

1. Emergence of max-plus approach. A system theory tailored for synchronization

This paper summarizes the history of max-plus algebra within the field of discrete event systems. It is based on brief survey of the role of max-plus algebra in the field of discrete event systems that appeared in Komenda, Lahaye, Boimond, and van den Boom (2017), but extended in several directions. In particular, there is a section, where computational aspects are discussed together with results about max-plus algebra from the computer science literature.

The emergence of a system theory for classes of discrete event systems (DES), in which max-plus algebra and similar algebraic tools play a central role, dates from the early 1980's. We emphasize that the idempotent semiring (also called dioid) of extended real numbers $(\mathbb{R} \cup \{-\infty\}, \max, +)$ is commonly called max-plus algebra, while it is not formally an algebra in the strictly mathematic sense.

Its inspiration stems certainly from the following observation: synchronization, which is a very non smooth and nonlinear phenomenon with regard to "usual" system theory, can be modeled by linear equations in particular algebraic structures such as max-plus algebra and other idempotent semiring structures (Cohen, Dubois, Quadrat, & Viot, 1983; Cuninghame-Green, 1979).

Two important features characterize this approach often called *max-plus linear system theory*:

- most of the contributions have used as a guideline the "classical" linear system theory;

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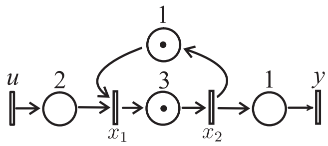

Fig. 1. A timed event graph.

- it is turned towards DES performance related issues (as opposed to logical aspects considered in other approaches such as automata and formal language theory) by including timing aspects in DES description.

A consideration has significantly contributed to the promotion and the scope definition of the approach: a class of ordinary[1] Petri nets, namely the timed event graphs (TEGs) has been identified to capture the class of stationary[2] max-plus linear systems (Cohen, Dubois, Quadrat, & Viot, 1985) and subsequent publications by Max Plus team.[3] TEGs are timed Petri nets in which each place has a single input transition and a single output transition. A single output transition means that no conflict is considered for the tokens consumption in the place, in other words, the attention is restricted to DES in which all potential conflicts have been solved by some predefined policy. Symmetrically, a single input transition implies that there is no competition in supplying tokens in the place. In the end, mostly synchronization phenomena (corresponding to the configuration in which a transition has several input places and/or several output places) can be considered, and this is the price to pay for linearity.

Example 1. Fig. 1 depicts a TEG, that is a Petri net in which each place (represented by a circle) has exactly one input transition (represented by a rectangle) and one output transition. The number next to a place indicates the sojourn time for a token, that is the number of units of time that must elapse before the token becomes available for the firing of the output transition. Let $u(k)$ denote the date of the k^{th} firing of transition u (same notation for x_1, x_2 and y). Considering the *earliest firing rule* (a transition is fired as soon as there is an available token in each input place), we have the following evolution equations

$$\begin{aligned} x_1(k) &= \max(2+u(k), 1+x_2(k-1)) \\ x_2(k) &= 3+x_1(k-1) \\ y(k) &= 1+x_2(k). \end{aligned}$$

Denoting $\oplus$ (resp. $\otimes$) the addition corresponding to max operation (resp. the multiplication corresponding to usual addition), we obtain linear equations in max-plus algebra, that is:

$$\begin{aligned} x_1(k) &= 2\otimes u(k)\oplus 1\otimes x_2(k-1) \\ x_2(k) &= 3\otimes x_1(k-1) \\ y(k) &= 1\otimes x_2(k) \end{aligned}$$

Rewriting the resulting equations in max-plus-algebraic matrix notation leads to a state-space representation:

$$\begin{cases} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \varepsilon & 1 \\ 3 & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} \oplus \begin{bmatrix} 2 \\ \varepsilon \end{bmatrix} \otimes u(k) \\ y(k) = \begin{bmatrix} \varepsilon & 1 \end{bmatrix} \otimes \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

where ε is equal to $-\infty$.

This new area of linear system theory has benefited from existing mathematical tools related to idempotent algebras such as lattice theory (Birkhoff, 1940), residuation theory (Blyth & Janowitz, 1972), graph theory (Gondran & Minoux, 1979), optimization (Zimmermann, 1981) and idempotent analysis (Kolokoltsov & Maslov, 1997), however it is worth mentioning that the progress has probably been impeded by the fact that some fundamental mathematical issues in this area are still open.

The overview of the contributions reveals that main concepts from linear system theory have been step by step specified into max-plus linear system theory. Without aiming to be exhaustive:

- several *possible representations* have been studied, namely state-space equations, transfer function in *event domain* (Cohen et al., 1983; 1985), *time domain* (Caspi & Halbwachs, 1986), and *two-dimensional domain* using series in two formal variables (Cohen, Moller, Quadrat, & Viot, 1986) (with more details in Cohen, Moller, Quadrat, & Viot, 1989);
- *performance analysis and stability* are mostly based on the interpretation of the *eigenvalue* of the state-matrix in terms of cycle-time, with its associated eigenspace and related cyclicity property (Baccelli, Cohen, Olsder, & Quadrat, 1992; Gaubert, 1997);
- a wide range of *control laws* have been adapted such as:
  - open-loop structures overcoming system output tracking (Baccelli et al., 1992, chap. 5.6), (Cofer & Garg, 1996; Menguy, Boimond, Hardouin, & Ferrier, 2000) or model reference tracking (Libeaut & Loiseau, 1996),
  - closed-loop structures taking into account disturbances and model-system mismatches (Cottenceau, Hardouin, Boimond, Ferrier et al., 1999; Lüders & Santos-Mendes, 2002), possibly including a state-observer (Hardouin, Maia, Cottenceau, & Lhommeau, 2010),
  - model predictive control scheme (De Schutter & van den Boom, 2001; van den Boom & De Schutter, 2002) with emphasis on stability in Necoara, De Schutter, van den Boom, and Hellendoorn (2007).

For a large survey on max-plus linear systems theory, we refer to books (Baccelli et al., 1992; Butkovič, 2010; Gunawardena, 1998; Heidergott, Olsder, & van der Woude, 2006), to manuscript (Gaubert, 1992) and surveys (Akian, Bapat, & Gaubert, 2003; Cohen, Gaubert, & Quadrat, 1999; Cohen et al., 1989; Gaubert, 1997).

2. Some extensions focused on synchronization in DES

There is an important connection between min-max-plus systems, in which time evolution depend on both max and min, but also addition operation and the game theory. It goes back to Olsder (1991), where spectral properties of such systems are studied. More recent references on this topic are Gunawardena (2003) and Akian, Gaubert, and Guterman (2012). The latter work establishes an equivalence with mean payoff games, an important open complexity problem in computer science, and it seems many verification problems for max-plus systems reduce to mean payoff games. We mention that many theoretical works on max-plus algebra and max-plus systems do not make use of the words “max-plus” but rather “tropical”. The adjective tropical was invented by French mathematicians, in honor of the Brazilian mathematician and computer scientist Imre Simon (1943–2009).

A natural generalization of deterministic max-plus-linear systems are stochastic max-plus-linear systems, which have been studied for more than two decades. Ergodic theory of stochastic timed event graphs is developed in Baccelli et al. (1992), where most of the theory is covered. In particular, asymptotic properties of stochastic max-plus-linear systems are studied therein in terms of the so-called Lyapunov exponents that correspond to the

[1] Petri nets in which all arc weights are 1.
[2] Stationarity is defined conventionally but over operators of max-plus algebra.
[3] Max Plus is a collective name for a working group on max-plus algebra, at INRIA Rocquencourt, comprising: Marianne Akian, Guy Cohen, Stéphane Gaubert, Jean-Pierre Quadrat and Michel Viot.

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