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Short review

Uniformly hyperbolic control theory

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ABSTRACT

This paper gives a summary of a body of work at the intersection of control theory and smooth nonlinear dynamics. The main idea is to transfer the concept of uniform hyperbolicity, central to the theory of smooth dynamical systems, to control-affine systems. Combining the strength of geometric control theory and the hyperbolic theory of dynamical systems, it is possible to deduce control-theoretic results of non-local nature that reveal remarkable analogies to the classical hyperbolic theory of dynamical systems. This includes results on controllability, robustness, and practical stabilizability in a networked control framework.

1. Introduction

The concept of uniform hyperbolicity, introduced in the 1960s by Stephen Smale, has become a cornerstone for the hyperbolic theory of dynamical systems, developed in the ensuing decades. This concept, which axiomatizes the geometric picture behind the horseshoe map and other complex systems, has been successfully generalized in various directions not long after its introduction to analyze a broad variety of systems (e.g., to non-uniform hyperbolicity, partial hyperbolicity and dominated splittings). A uniformly hyperbolic (discrete-time) system is essentially characterized by the fact that the linearization along any of its orbits behaves like a linear operator without eigenvalues on the unit circle, i.e., by a splitting of each tangent space into a direct sum of a stable and an unstable ‘eigenspace’. The uniformity is expressed by a uniform estimate on the contraction and expansion rates. We refer to Hasselblatt (2002) for a comprehensive survey of results related to hyperbolic dynamical systems.

Uniform hyperbolicity and its generalizations also occur quite naturally in nonlinear control systems, which calls for a systematic transfer of the methods developed for the analysis of hyperbolic dynamical systems to control systems in order to gain new insights in control-theoretic problems. However, so far not much effort has been put into the development of a ‘hyperbolic control theory’. The aim of this paper is to provide a survey of the existing results, which show that a combination of techniques from geometric control theory and the uniformly hyperbolic theory of dynamical systems can lead to deep insights about global and semiglobal properties of control-affine systems with a compact and convex control range.

These results are grounded on the topological theory of Colonius and Kliemann (2000) which provides an approach to understanding the global controllability structure of control systems. Two central notions of this theory are control and chain control sets. Control sets are the maximal regions of complete approximate controllability in the state space. The definition of chain control sets involves the concept of ε-chains (also called ε-pseudo-orbits) from the theory of dynamical systems. The main motivation for the concept of chain control sets comes from the facts that (i) chain control sets are outer approximations of control sets and (ii) chain control sets in general are easier to determine than control sets (both analytically and numerically).

As examples show, chain control sets can support uniformly hyperbolic and, more generally, partially hyperbolic structures. For instance, every chain control set of a control-affine system on a flag manifold of a noncompact real semisimple Lie group, induced by a right-invariant system on the group, admits a partially hyperbolic structure, i.e., an invariant splitting of the tangent bundle into a stable, an unstable and a central subbundle. The paper (Da Silva & Kawan, 2016b) provides a complete classification of those chain control sets on flag manifolds which are uniformly hyperbolic, using extensively the semigroup theory developed by San Martin and co-workers (Barros & San Martin, 2007; San Martin, 1998; San Martin & Seco, 2010; San Martin & Tonelli, 1995). Another way how a uniformly hyperbolic chain control set can arise is

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by adding sufficiently small control terms to an uncontrolled equation with a uniformly hyperbolic chain transitive set. In this case, the uniformly hyperbolic invariant set blows up to a uniformly hyperbolic chain control set, cf., Colonius and Du (2001).

In the case of a uniformly hyperbolic chain control set, tools from the theory of smooth dynamical systems have been be applied to analyze controllability and robustness properties. In particular, it has been proved that a uniformly hyperbolic chain control set is the closure of a control set under the assumption of local accessibility, cf. Colonius and Du (2001). As a consequence, complete controllability holds on the interior of the chain control set and the chain control set varies continuously in the Hausdorff metric in dependence on system parameters.

Another control application of uniformly hyperbolic theory concerns the problem of practical stabilization under information constraints. Stabilization problems involving a communication channel of finite capacity which provides the controller with state information, have been studied by many authors (see, e.g., the survey (Nair, Fagnani, Zampieri, & Evans, 2007) and the monographs (Kawan, 2013; Matveev & Savkin, 2009; Yüksel & Başar, 2013)). The main theoretical problem here is to determine the smallest capacity above which the stabilization objective can be achieved. For practical stabilization in the sense of rendering a compact subset Q of the state space invariant, the notion of invariance entropy $h_{\text{inv}}(Q)$ was introduced in Colonius and Kawan (2009) as a measure for the associated critical channel capacity. This quantity measures the exponential complexity of the control task of keeping the system inside Q. In Da Silva and Kawan (2016c) a formula for the invariance entropy $h_{\text{inv}}(Q)$ of a uniformly hyperbolic chain control set Q has been derived in terms of unstable volume growth rates along trajectories in Q. The proof of this formula in particular reveals the interesting fact that in order to make Q invariant with a capacity arbitrarily close to $h_{\text{inv}}(Q)$, control strategies that stabilize a periodic orbit in Q are as good as any other strategy, thus this class of strategies is optimal.

The paper (Colonius & Lettau, 2016) gives an application of this result to a problem related with a continuously stirred tank reactor. Moreover, the paper (Da & Kawan, 2016a) shows that the invariance entropy of uniformly hyperbolic chain control sets depends continuously on system parameters.

In the following Sections 2–7, we explain these results in greater detail. In Section 8, we give a brief account of the related subjects known as 'control of chaos' and 'partial chaos', and in Section 9 we outline some problems and ideas for future research.

Notation: We write cl$A$ and int$A$ for the closure and the interior of a set A, respectively. If M is a smooth manifold, we write T_xM for the tangent space to M at x, and TM for the tangent bundle of M. If $f: M \to N$ is a smooth map between manifolds, $\mathrm{D}f(x): T_xM \to T_{f(x)}N$ denotes its derivative at $x \in M$.

2. Control sets and chain control sets

A control-affine system is governed by differential equations of the form

$$\Sigma: \ \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \ u \in \mathcal{U}, \tag{1}$$

where $x(t)$ lives on a Riemannian manifold M (the state space) and $\mathcal{U}$ is the set of admissible control functions, which we assume to be of the form $\mathcal{U} = L^\infty(\mathbb{R}, U)$ with $U \subset \mathbb{R}^m$ being a compact and convex set with $0 \in \text{int}U$. Assuming that $f_0, f_1, \dots, f_m$ are $\mathcal{C}^1$-vector fields and that the unique solution $\varphi(t, x, u)$ for the initial value x at time $t_0 = 0$ and the control u exists for all $t \in \mathbb{R}$, regardless of $(u, x) \in \mathcal{U} \times M$, we obtain a skew-product flow (i.e., a flow of triangular structure)

$$\Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \ t \in \mathbb{R},$$

that acts on the extended state space $\mathcal{U} \times M$. Here

$$\theta_t u = u(t + \cdot), \quad \theta_t : \mathcal{U} \to \mathcal{U}, \ t \in \mathbb{R},$$

denotes the shift flow on $\mathcal{U}$. With the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$, $\mathcal{U}$ becomes a compact metrizable space and Φ a continuous flow, called the *control flow* of Σ. We write $\varphi_{t,u} = \varphi(t, \cdot, u)$.

A *control set* of Σ is a subset $D \subset M$ such that

(i) for every $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, u) \subset D$,
(ii) for all $x, y \in D$ and every neighborhood N of y there are $u \in \mathcal{U}$ and $T > 0$ with $\varphi(T, x, u) \in N$ (i.e., approximate controllability holds on D), and
(iii) D is maximal with (i) and (ii) in the sense of set inclusion.

A *chain control set* $E \subset M$ is a set such that

(i) for every $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset E$,
(ii) for all $x, y \in E$ and $\varepsilon, T > 0$ there are $n \in \mathbb{N}$, $u_0, \dots, u_{n-1} \in \mathcal{U}$, $x = x_0, x_1, \dots, x_n = y$ and $t_0, t_1, \dots, t_{n-1} \geq T$ such that $d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, \dots, n-1$, and
(iii) E is maximal with (i) and (ii) in the sense of set inclusion.

Before we proceed, for the convenience of the reader, we explain the concept of chain transitivity used in the topological theory of dynamical systems to analyze recurrence properties (see also Colonius and Kliemann (2000, Appendix B)). If $\phi : \mathbb{R} \times X \to X$ is a continuous flow on a metric space (X, d), a set $A \subset X$ is called *chain transitive* if for all $x, y \in A$ and $\varepsilon, T > 0$ there exists an (ε, T)-*chain* from x to y, i.e., there are $n \in \mathbb{N}$, points $x = x_0, x_1, \dots, x_n = y$, and times $t_0, t_1, \dots, t_{n-1} \geq T$ so that $d(\phi(t_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, \dots, n-1$. A point $x \in X$ is called *chain recurrent* if for all ε, $T > 0$ there is an (ε, T)-chain from x to x. If X is compact, then the set $R(\phi)$ of all chain recurrent points is closed and invariant. Moreover, the connected components of $R(\phi)$ are precisely the maximal invariant chain transitive sets and are called *chain recurrent components*. The chain recurrent set essentially contains all relevant dynamical information of the flow. For instance, all α- and ω-limit set are contained in $R(\phi)$.

Now we consider again the control-affine system (1). The *lift* of a chain control set E is defined by

$$\mathcal{E} := \{(u, x) \in \mathcal{U} \times M \ : \ \varphi(\mathbb{R}, x, u) \subset E\}.$$

It is a maximal invariant chain transitive set of the control flow, hence a chain recurrent component if M is compact. If Σ is locally accessible and D is a control set with nonempty interior, then D is contained in a chain control set (which is unique, since different chain control sets are disjoint). In general, chain control sets are closed, while control sets are neither open nor closed except when they are invariant in backward or forward time, respectively.

A chain control set E is *uniformly hyperbolic without center bundle* if it is compact and for every $(u, x) \in \mathcal{E}$ there exists a splitting

$$T_xM = E^-_{u,x} \oplus E^+_{u,x}$$

into linear subspaces such that

(i) $\mathrm{D}\varphi_{t,u}(x)E^\pm_{u,x} = E^\pm_{\Phi_t(u,x)}$ for all $t \in \mathbb{R}$ and $(u, x) \in \mathcal{E}$, and
(ii) there are constants $c, \lambda > 0$ such that for all $(u; x) \in \varepsilon$,

$$|\mathrm{D}\varphi_{t,u}(x)v| \leq c^{-1}\mathrm{e}^{-\lambda t}|v| \quad \text{for all } t \geq 0, \ v \in E^-_{u,x}$$

and

$$|\mathrm{D}\varphi_{t,u}(x)v| \geq c\mathrm{e}^{\lambda t}|v| \quad \text{for all } t \geq 0, \ v \in E^+_{u,x}.$$

This definition is independent of the Riemannian metric, however, the constant c depends on the choice of the metric. From the two conditions it automatically follows that the subspaces $E^\pm_{u,x}$ change continuously with (u, x), cf. Kawan (2013, Chapter 6).