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Some new applications of Russell's principle to infinite dimensional vibrating systems

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The aim of this work is to highlight the interest of a by now classical methodology, commonly called *Russell's principle*, in proposing new control strategies and estimates for infinite dimensional vibrating systems. After describing (with complete proofs) a particular version, of interest for our work, of Russell's principle, we consider two main applications. The first one, which mainly contains results which are new, studies the approximation of a class of boundary control systems by systems with controls distributed in an open set (internal controls), with the support shrinking to the boundary. These approximations are interesting since for the approximating systems we have bounded input operators, which makes easier the use of many control theoretic tools. The second application concerns the approximation of exact controls for infinite dimensional systems using their projections on finite dimensional spaces. We propose here an alternative, based on Russell's principle, of the existing approximation methods, often based on inverting the Gramian.

1. Introduction

The original version of Russell's principle says, roughly speaking that forwards and backwards exponential stabilizability implies exact controllability. The first application to systems governed by PDEs has been given by Russell himself in Russell (1978, Section 5), where he established the exact controllability for a wave equation with boundary control. This approach has been subsequently used to prove exact controllability for infinite dimensional systems by many authors, see for instance Chen (1979) and Komornik (1994). An abstract version of this principle has been given in Chen (1979), in the case of bounded control operators. This version has been further generalized in Rebarber and Weiss (1997) and Natarajan and Weiss (2013). More recently, a dual version Russell's principle, asserting that forward and backward detectability implies observability, came to the attention of control theorists. They developed, in particular, the concept of back and forth observers, which has been been proposed in Shim, Tanwani, and Ping (2012) for finite dimensional, possibly nonlinear systems and in Ramdani, Tucsnak, and Weiss (2010) for linear infinite dimensional systems (see also Ito, Ito, Ramdani, & Tucsnak (2011)).

The purpose of this work is to show that Russell's principle can be successfully adapted for a class of infinite dimensional systems involving perturbations and approximations. Intuitively, the exact controls constructed via this principle seem less "oscillating" than those obtained by inverting the Gramian (also designed, following Lions (1988), by *HUM method*), which makes them more robust in view of perturbations and approximations. We focuss on two classes of applications.

The first one allows obtaining new results on a class of singular perturbation methods which have a geometric nature. More precisely, we want to understand the convergence of exact controls and of the associated state trajectories for a class of distributed control systems towards the corresponding objects for a boundary control system. The methodology is first developed in an abstract setting and then applied to families of systems governed by the string equation, with controls supported in a family of intervals shrinking towards an extremity of the string. As far as we know, the only papers to study this phenomena are Fabre

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(Fabre, 1992; Fabre & Puel, 1994), Fabre and Puel (1993) and Joly (2006). The main results in Fabre (1992); Fabre and Puel (1993); 1994) provide sufficient conditions for the weak convergence of the minimal L^2 in time norm controls (often referred to as *HUM controls*) and of the corresponding trajectories towards those associated to the limiting control system. Joly in Joly (2006) investigated, in both the linear and nonlinear contexts, the strong convergence of solutions of the wave equation with locally distributed damping towards solutions of the wave equation with boundary damping. Unlike Fabre (1992); Fabre and Puel (1993); 1994), where minimal norm controls were considered, in this work we utilize controls generated using Russell's principle. This approach allows us to obtain a new abstract result that includes some cases of interest for the wave equation in one space dimension with stronger convergence properties. An important part of the effort in this part is devoted to obtaining wellposedness and observability estimates for the string equation which are independent of the size of the control interval (this requires an appropriate scaling of the control operator).

The second class of applications we are interested in is analyzing the approximation by finite dimensional systems, in particular giving error estimates, for exact controls for infinite dimensional vibrating systems. As a first step in this procedure we give a Russell's principle based construction of smooth controls for smooth initial states. We next consider the natural problem of approximating exact controls for infinite dimensional vibrating systems by controls associated to the projections of the considered systems on appropriate families of finite dimensional spaces. The work in this direction has been highly developed since the 90′s, following a series of papers of Glowinski and Lions (see Glowinski, Li, & Lions, 1990; Glowinski & Lions, 1996) where algorithms to determine the minimal L^2-norm exact controls (sometimes called HUM controls) are provided. Several abnormalities presented in these works stand at the origin of a large number of articles in which a great variety of numerical methods are presented and analyzed (see, for instance, Ervedoza and Zuazua, 2009; Zuazua, 2005 and the references therein). Much less is done concerning the rate of convergence of the approximations of these controls. In the case of HUM controls for the one dimensional wave equation, as far as we know, the only result in this direction has been obtained in Ervedoza and Zuazua (2010). In the last part of this work we describe, following Cîndea, Micu, and Tucsnak (2011), a Russell's principle based numerical method for computing exact controls for a class of infinite dimensional systems modelling elastic vibrations. Our main theoretical result gives the rate of convergence of our approximations to an exact control.

The remaining part of this work is organized as follows. In Section 2 we remind the derivation of Russell's principle in the particular case of systems governed by second order differential equations in a Hilbert space, with a bounded control operator. Some preliminary background results are provided in Section 3. Section 4 is devoted to an abstract singular perturbation result, which is given in Theorem 4.1. The proof that the ε-problems for the string with homogeneous Neumann boundary condition converge to the Neumann control problem (5.4)–(5.6) is given in Section 5 by checking that these systems fit the hypothesis of our general abstract result in Theorem 4.1. A similar procedure is used in Section 6 for the ε-problems for the string with homogeneous Dirichlet conditions and the limiting Dirichlet control problem (6.1)–(6.3). Finally, in Section 7 we describe a Russell's principle based strategy to approximate the exact controls for infinite dimensional vibrating systems using finite dimensional control systems and we provide some error estimates.

2. The case of bounded input operators and construction of smooth controls

For reader's convenience, we briefly recall below Russell's construction in the particular case of a second order linear differential equation in a Hilbert space and, to avoid technicalities, in the case of a bounded control operator. To this aim, let $\mathcal{H}$ be a Hilbert space. The inner product on $\mathcal{H}$ is denoted, for the remaining part of this paper, by $\langle \cdot, \cdot\rangle$ and the associated norm by $\|\cdot\|$. Let $\mathcal{U}$ be another Hilbert space. Throughout this work $\mathcal{H}$ and $\mathcal{U}$ will be identified with their duals and they will be used as pivot spaces when specifying the adjoints of various linear operators. Assume that $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ is a self-adjoint, strictly positive operator with compact resolvents. Then, according to classical results, the operator A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k\geq 1}$ of eigenvectors and the corresponding family of positive eigenvalues $(\lambda_k)_{k\geq 1}$ satisfies $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, we have

$$\mathcal{D}(A_0) = \left\{ z \in \mathcal{H} \;\middle|\; \sum_{k\geqslant 1} \lambda_k^2 |\langle z, \varphi_k\rangle|^2 < \infty \right\},$$

and

$$A_0 z = \sum_{k\geqslant 1} \lambda_k \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A_0)).$$

For $\alpha \geq 0$ the operator A_0^α is defined by

$$\mathcal{D}(A_0^\alpha) = \left\{ z \in \mathcal{H} \;\middle|\; \sum_{k\geqslant 1} \lambda_k^{2\alpha} |\langle z, \varphi_k\rangle|^2 < \infty \right\}, \tag{2.1}$$

and

$$A_0^\alpha z = \sum_{k\geqslant 1} \lambda_k^\alpha \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A_0^\alpha)).$$

For every $\alpha \geq 0$ we denote by $\mathcal{H}_\alpha$ the space $\mathcal{D}(A_0^\alpha)$ endowed with the inner product

$$\langle \varphi, \psi\rangle_\alpha = \langle A_0^\alpha \varphi, A_0^\alpha \psi\rangle \qquad (\varphi,\ \psi \in H_\alpha),$$

and induced norm denoted by $\|\cdot\|_\alpha$. To be coherent with the notation above, we simply write $\langle \cdot, \cdot\rangle$ for $\langle \cdot, \cdot\rangle_0$ and $\|\cdot\|$ for $\|\cdot\|_0$. From the above facts it follows that for every $\alpha \geq 0$ the operator A_0 is a unitary operator from $\mathcal{H}_{\alpha+1}$ onto $\mathcal{H}_\alpha$ and A_0 is strictly positive on $\mathcal{H}_\alpha$.

Let $B_0 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ be an input operator. Consider the system

$$\ddot{q}(t) + A_0 q(t) + B_0 u(t) = 0 \ (t \geq 0), \tag{2.2}$$

$$q(0) = q_0, \ \dot{q}(0) = q_1. \tag{2.3}$$

It is by now routine that for every $u \in L^2([0,\infty); \mathcal{U})$ the above equations admit a unique solution

$$w \in C([0,\infty); \mathcal{H}_{\frac{1}{2}}) \cap C^1([0,\infty); \mathcal{H}),$$

which satisfies, for every $u \in L^2([0,\infty); \mathcal{U})$ and $t \geq 0$, the *energy estimate*:

$$\frac{1}{2}\left(\|\dot{w}(0)\|^2 + \|w(0)\|_{\frac{1}{2}}^2 \right) - \frac{1}{2}\left(\|\dot{w}(t)\|^2 + \|w(t)\|_{\frac{1}{2}}^2 \right) = \int_0^t \langle u(\sigma), B_0^* \dot{w}(\sigma)\rangle_{\mathcal{U}}\, \mathrm{d}\sigma. \tag{2.4}$$

In other terms, Eqs. (2.2)–(2.3) define a well-posed control system with input space $\mathcal{U}$ and state space $X = \mathcal{H}_{\frac{1}{2}} \times \mathcal{H}$.

The above system is said *exactly controllable in time* $\tau > 0$ if for every $q_0 \in \mathcal{H}_{\frac{1}{2}}$, $q_1 \in \mathcal{H}$ there exists a control $u \in L^2([0,\tau], \mathcal{U})$ such that $q(\tau) = 0$ and $\dot{q}(\tau) = 0$.

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