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Large time control and turnpike properties for wave equations

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ABSTRACT

In the last decades mathematical control theory has been extensively developed to handle various models, including Ordinary and Partial Differential Equations (ODE and PDE), both of deterministic and stochastic nature, discrete and hybrid systems.

However, little attention has been paid to the length of the time horizon of control, which is necessarily long in many applications, and to how it affects the nature of controls and controlled trajectories. The *turnpike property* refers precisely to those aspects and stresses the fact that, often, optimal controls and trajectories, in long time intervals, undergo some relevant asymptotic simplification property ensuring that, during most of the time-horizon of control, optimal pairs remain close to the steady-state optimal one.

Due to the intrinsic finite velocity of propagation and the oscillatory nature of solutions of the free wave equation, optimal controls for waves are typically of oscillatory nature. But, despite this, as we shall see, under suitable coercivity conditions on the cost functional to be minimised and when controllability holds, the turnpike property is also fulfilled for the wave equation.

When this occurs, the approximation of the time-depending control problem by the steady-state one is justified, a fact that is often employed in applications to reduce the computational cost.

We present some recent results of this nature for the wave equation and other closely related conservative systems, and discuss some other related issues and a number of relevant open problems that arise in this field.

1. Problem formulation

Optimal control problems play a key role in many fields and applications to industry, technology and other sciences. In the last decades the mathematical theory has been extensively developed to handle these problems for various models, including Ordinary and Partial Differential Equations (ODE and PDE), both of deterministic and stochastic nature, discrete and hybrid systems.

The existing theory provides systematic methods to prove the existence of optimal controls, characterise them through optimality conditions, build feedback controllers and efficient computational methods. Often times, however, little attention is paid to the length of the time horizon of control, which is necessarily long in many applications, and how it affects the nature of controls and controlled trajectories.

But, dealing with long time intervals is sometimes necessary to face specific applications and it has important consequences on the nature of the controls and increases the computational cost significantly, becoming often prohibitive. It is therefore natural to develop specific tools conceived to deal with control problems in long horizons of time, adapted to its specific nature and structure.

In this paper we summarise the recent work of our team in this field focusing on the paradigmatic example of the wave equation, and establish links between the property of *turnpike* and the classical notion of controllability.

Turnpike refers to the fact that optimal controls and trajectories, in long time intervals, undergo some relevant, and somehow unexpected, asymptotic simplification property ensuring that, during most of the time-horizon of control, optimal pairs remain close to the steady-state optimal one, the turnpike state.

Due to the intrinsic finite velocity of propagation and the oscillatory nature of solutions of the free wave equation, optimal controls for waves are typically of oscillatory nature. Thus, the fullfilement of the turnpike property might seem surprising in a

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first glance, although it is not really so since it is closely related to the exponential decay of solutions of the Linear Quadratic Regulator Problems (LQR) in infinite horizons of time.

Despite of this, as we shall see, under suitable coercivity conditions on the cost functional to be minimised, when controllability holds, the turnpike property is also fulfilled.

When this occurs, the approximation of the time-depending control problem by the steady-state one, a fact that is often employed in applications to reduce the computational cost, is justified.

We present some recent results of this nature for the wave equation and discuss some other related issues and a number of relevant open problems that arise in this field.

To do it on a concrete model example we mainly consider the wave equation

$$\begin{cases} y_{tt} - \Delta y = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0,\ y_t(0) = y_1, \end{cases} \tag{1.1}$$

in a bounded domain Ω of the d-dimensional Euclidean space, with a control function $u = u(x,t)$ acting on an open non-empty subset ω of Ω during the time horizon (0, T).

This is one of the most paradigmatic examples of control problem for an infinite-dimensional conservative model, the wave equation, arising in a variety of applied contexts. The state $y = y(x,t)$ may represent an acoustic signal or the deformation of some flexible membrane, and the control $u = u(x,t)$, whose action is localised in ω, letting vibrations to propagate freely outside ω, models some exterior applied source or force.

In this article we shall mainly focus on this model but most of the contents apply in a much broader context of conservative infinite-dimensional dynamics. We refer to Porretta and Zuazua (2013) for a general discussion of the corresponding abstract semigroup setting.

When $(y_0,y_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $u \in L^2(\Omega\times(0,T))$ system (1.1) has a unique finite energy solution $y\in C\big([0,T];H_0^1(\Omega)\big)\cap C^1\big([0,T];L^2(\Omega)\big)$.

To motivate the problem under consideration and discriminate the various scenarios in which the *turnpike* phenomenon emerges or not, we begin considering the classical controllability problem where the control is aimed to drive the state to a given target in the final time $t = T$, paying special attention to the behaviour of controls and controlled dynamics in long time intervals.

Often in practice, when the free dynamics of the system under consideration enjoys some stability property so that, for instance, time-evolving solutions tend to steady-state ones as $t\to\infty$, it is expected that time evolving controls will also reproduce that property when the time control horizon is long enough. This is the so-called *turnpike property* and it is very much in agreement with intuition, in particular, for parabolic-like problems, where the inner dissipative mechanisms are likely to force the desired stability property of controls and controlled dynamics.

But whether this behaviour is still to be expected for wave-like models, where solutions of the free dynamics are of oscillatory nature, conserve energy and, therefore, do not enjoy the property of asymptotic simplification, is less clear.

In this paper we shall however show, collecting previous earlier results, that the turnpike property still holds for wave-like models under suitable controllability assumptions. This fact is relevant since even if, a priori, one is simply interested in other kinds of optimal control problems, such as minimising a quadratic cost functional, without paying attention to the terminal conditions at the final time $t = T$, the emergence of the turnpike phenomenon is very tightly connected with the fulfilment of the controllability property.

These results are relevant and find application in various different ways:

1. The turnpike property ensures the asymptotic simplification of controls and controlled trajectories for optimal control problems involving wave-like models in long-time horizons, so that optimal pairs are close to the steady-state ones during most of the time interval. More precisely, except for an initial time layer $[0,\tau]$ and a final one $[0,T-\tau]$, in which the controlled dynamics has to match the initial and terminal conditions, during the rest of the time interval $[\tau,T-\tau]$ the control and controlled trajectories are exponentially close to the steady-state ones.
2. This property can be used as test for the accuracy of the numerical simulation codes: those not reproducing the property of turnpike could be considered as unsuitable ones.
3. This asymptotic simplification property can serve also to initialise iterative methods for solving the optimality system characterising optimal pairs, employing the steady-state optimal pairs, which are cheaper to compute, in the initialisation step.
4. The turnpike state also serves as an initialisation to Receding Optimal Control (ROC) or Model Predictive Control (MPC) methods (see Grüne, 2013; 2016).

As we shall see in the present paper, even when dealing with classical Linear-Quadratic (LQ) optimal control problems, for the turnpike property to hold two ingredients will be needed:

1. The system under consideration needs to enjoy the property of controllability. And this is so even if, a priori, one is not interested in controllability issues. But for the turnpike property to hold, ensuring in particular that the optimal trajectory stays most of the time near the turnpike point, one needs that the action of the control suffices to ensure the controllability property.
2. The cost functional needs to be sufficiently coercive. In other words, it needs to penalise the control but also the state sufficiently so that the partial information of the state involved in the cost functional suffices to get complete information in the full state. This is a observability property that, for wave-like equations, requires some geometric restrictions on the subdomain where the control is being applied and a sufficiently long control time-horizon. The later is not an issue when analysing turnpike properties since the time-horizon tends to infinity. But the former needs to be taken into account with care, imposing suitable geometric conditions on the support of the control.

In order to have a complete understanding of the turnpike property one needs to carefully analyse the behaviour of the Optimality System (OS) characterising optimal controls and trajectories. It is constituted by a coupled system of two wave equations, one evolving in the forward sense of time while the other one does it in the backward sense. State and control are fully coupled through this system. Accordingly, the turnpike property needs to hold in both variables (state and adjoint state or co-state) simultaneously.

The two conditions under which the turnpike property mentioned above holds play dual and complementary roles in what concerns the behaviour of the OS to ensure the turnpike property. The controllability of the state equation can be understood as an observability property on the adjoint. On the other hand, the fact that the cost functional involves enough information on the state can be understood as an observability property of the state equation and thus, a controllability one for the adjoint system.

Overall, under these assumptions, the turnpike property is fulfilled and during most of the time-horizon [0, T] optimal controls and trajectories are exponentially close to the steady state one. The later ones are, of course, much easier to compute since they are determined as the solution of an optimal control problem for