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IFAC-PapersOnLine 49-8 (2016) 001–006

Lyapunov-Based Error Bounds for the Reduced-Basis Method

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Abstract: We introduce two new error bounds for the reduced-basis method. Existing error bounds for parabolic problems can be classified as either space time or time stepping. Space-time bounds are much more costly and often become unpractical. The cheaper times-stepping bounds have always failed to adequately represent the dynamics of systems containing noncoercive operators. As a result they have always produced extremely pessimistic bounds. Our new bounds are time-stepping bounds that make use of the Lyapunov stability theory to better capture the dynamics of the system.

Keywords: Error Estimation, Reduced-Order Models, Lyapunov Function, Lyapunov Equation, Stability Analysis, Parameter-Dependent Systems, Dynamic Systems

1. INTRODUCTION

Reduced-basis modeling is a powerful tool for approximating solutions of parameter-dependent PDEs. In recent years it has been applied to many types of PDEs allowing for drastic decreases in computational time. As with all model-reduction techniques, error quantification is of great importance. If the error is too large, the model will not be useful. Error bounds also play a second role in reduced-basis modeling; they are used in the construction of the model itself.

Two classical error bounds that can be used in conjunction with the reduced-basis method for parabolic equations are the energy-error estimate given by Grepl and Patera (2005), and the L^2-error bound given by Haasdonk and Ohlberger (2008). Both are mainly for use with coercive problems. If they are extended for use with noncoercive problems, they predict exponential growth of the error in time. In such cases they are of little practical use.

The problem is that these bounds are based on how the system evolves in a single time step. This is convenient because it allows us to take complete advantage of the evolutionary nature of the system but has resulted in very pessimistic error bounds because the norms that have been used fail to capture the nature of the dynamics. An alternative approach is to use bounds based on the space-time formulation given by Urban and Patera (2014). That has been shown to be very successful in producing more accurate error bounds but is associated with greatly increased computational costs. Such bounds fail to take advantage of the evolutionary nature of the system and require the calculation of stability constants for the full space-time system.

In this article we present new generalizations of the time-stepping bounds. Our new bounds take full advantage of the evolutionary nature of the problem making them much cheaper than space-time bounds. At the same time they also model the dynamics of the system better so that the bounds do not grow exponentially. This is achieved using the Lyapunov stability theory and norms in which the error is well behaved.

For a large number of noncoercive problems our new bounds will produce better results than any previous bounds. They could also be highly useful in simulating closed-loop systems, which are often noncoercive.

2. PROBLEM SETUP AND ASSUMPTIONS

Most applications of our work will involve the approximation of PDEs but we will start directly with a semi-discrete system, which could be a discretization of a PDE. We will consider the problem of approximating the following system for any parameter μ in a bounded, finite-dimensional domain $\mathcal{D} \subset \mathbb{R}^p$.

$$M(\mu)\dot{y}(t) + A(\mu)y(t) = B(\mu)u(t) \quad (1)$$

Here we have $M(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, $A(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, $B(\mu) \in \mathbb{R}^{\mathcal{N}\times m}$, the inputs $u \in \mathbb{R}^m$, and the state $y(t) \in \mathbb{R}^{\mathcal{N}}$. We will write $\dot{y}$ to denote the time derivative of y and choose a symmetric positive definite (SPD) matrix $X \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ that we will use as an inner product. In section 6 we will assume that $M(\mu)$ be symmetric but most of our results will hold even if it is not.

In order to approximate solutions to (1) numerically we will use a backwards-Euler time discretization with K time steps of uniform length τ to get the following system

$$M(\mu)y^k + \tau A(\mu)y^k = M(\mu)y^{k-1} + \tau B(\mu)u^k, \quad (2)$$

for $1 \leq k \leq K$ which will be referred to as our truth system. For simplicity we will assume that the initial state y^0 is zero.

As is usual in the reduced-basis context, we will be interested in approximating y for any μ in a bounded parameter

10.1016/j.ifacol.2016.07.409

domain $\mathcal{D}$. In order to ensure that our method will be computationally efficient we require that the parameter-dependence of M, A, and B be of the following form

$$G(\mu) = \sum_{q=1}^{Q_G} \Theta_G^q(\mu) G^q, \tag{3}$$

where G should be replaced by A, M or B respectively. Here the parameter-independent components G^q will have the same size as the associated matrix G and the parameter maps $\Theta_G^q(\cdot)$ can be arbitrary smooth functions.

In later sections we will often suppress the μ-dependence of the matrices to save space and simplify the notation.

3. THE REDUCED-BASIS METHOD FOR PARABOLIC PROBLEMS

Following the work of Grepl and Patera (2005) we will use a Galerkin projection to reduce the truth model. We start by introducing a matrix $Z \in \mathbb{R}^{\mathcal{N}\times N}$, where $N \ll \mathcal{N}$. The columns of Z will be our N basis elements and should be orthonormal with respect to X. We will approximate our truth model with the reduced-basis model

$$M_N(\mu)y_N^k + \tau A_N(\mu)y_N^k = M_N(\mu)y_N^{k-1} + \tau B_N(\mu)u^k, \tag{4}$$

for $1 \le k \le K$, where $M_N := Z^TMZ$, $A_N := Z^TAZ$, $B_N := Z^TB$, and $y_N^0 = 0$.

In previous time-stepping methods the coercivity constant

$$\alpha(\mu) := \inf_{v\neq 0} \frac{v^TA(\mu)v}{v^TXv} \tag{5}$$

played a vital role in both establishing the stability of reduced-basis approximations and error estimation. The operator $A(\mu)$ is said to be coercive if $\alpha(\mu) > 0$. Grepl and Patera (2005) assume that both $A(\mu)$ and $M(\mu)$ are uniformly coercive and that $M(\mu)$ is symmetric for all $\mu \in \mathcal{D}$. In that case it is easy to show that (4) is stable both numerically and in the sense of Lyapunov for all choices of Z. Our relaxed assumptions do not suffice to guarantee stability but in section 7 we discuss ways in which stability can be ensured.

3.1 Building the Reduced Basis

Over the years many methods have been developed to build the reduced basis, onto which a system will be projected. In general the most effective method seems to be the POD-greedy method introduced by Haasdonk and Ohlberger (2008). The method is based on the greedy method that was used by Veroy et al. (2003) for stationary problems. In both cases the key idea is that a search is performed over a finite subset of $\mathcal{D}$ to find parameter values for which the model produces large error bounds. The model can then be improved using the truth solution associated with those parameter values.

In some cases the input u to the system will be parameter dependent or constant. In that case the input can be handled directly by the POD-greedy algorithm. If that is not the case, it may be useful to handle the input using impulse responses as done by Grepl and Patera (2005).

For our work the traditional methods will in many cases suffice but this is not guaranteed. Even if the truth system is stable, the reduced model can have stability issues. Such issues will be discussed in section 7.

4. LYAPUNOV STABILITY FOR LTI SYSTEMS

We will now introduce Lyapunov stability, which, as we will show, can be used to take advantage of the structure of the dynamical system and build cheap and effective error bounds.

4.1 Continuous-Time Systems

For continuous-time systems we have the following classical theorem.

Theorem 1. Given a fixed SPD matrix $P \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ the function $V(v) = v^TM(\mu)^TPM(\mu)v$ is a Lyapunov function for (1) at the parameter value $\mu \in \mathcal{D}$, iff the symmetric matrix

$$Q(\mu) := \frac{A(\mu)^TPM(\mu) + M(\mu)^TPA(\mu)}{2} \tag{6}$$

is positive definite. In that case the system is asymptotically stable for μ.

A sufficient condition to show that (1) is asymptotically stable for a given parameter value $\mu \in \mathcal{D}$ is that the coercivity constant

$$\alpha_Q(\mu) := \inf_{v\neq 0} \frac{v^TQ(\mu)v}{v^TXv}. \tag{7}$$

be positive. In that case we will say that $Q(\mu)$ is coercive.

4.2 Discrete-Time Systems

For the discrete-time system (2) we have the following theorem.

Theorem 2. Given a fixed SPD matrix $P \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ the function $V_D(v) = v^T(M(\mu)+\tau A(\mu))^TP(M(\mu)+\tau A(\mu))v$ is a Lyapunov function for (2) at the parameter value $\mu \in \mathcal{D}$, iff the symmetric matrix

$$Q_D(\mu) := \frac{A^TPM + M^TPA + \tau A^TPA}{2}. \tag{8}$$

is positive definite. In that case the system is asymptotically stable for μ.

We note that $Q_D - Q$ is semi-positive definite. That implies that any P that proves the stability of (1) also proves the stability of (2). For Q_D we introduce the coercivity constant α_{Q_D}, which is defined analogous to α_Q.

5. LYAPUNOV-BASED ERROR BOUNDS

In this section we derive generalized versions of the energy and the L_2 error bounds using Lyapunov stability theory. In both cases the error that we wish to measure will be given by $e^k = y^k - Zy_N^k$. We note that y_N^k does not need to be a reduced-basis approximation and could be any lower-dimensional approximation of y^k.