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Boundary control of coupled reaction-advection-diffusion equations having the same diffusivity parameter ⋆

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Abstract: We consider the problem of boundary stabilization for a system of n coupled parabolic linear PDEs of the reaction-diffusion-advection type. Particularly, we design a state-feedback law with Dirichlet-type actuation on only one end of the domain and prove exponential stability of the closed-loop system with an arbitrarily fast convergence rate. The backstepping method is used for controller design, and the transformation kernel matrix is derived by using the method of successive approximations to solve the corresponding PDE. Simulation results support the effectiveness of the suggested design.

Keywords: Distributed-parameter systems; Reaction-advection-diffusion equation; Boundary Control; Backstepping

1. INTRODUCTION

In this paper, the boundary stabilization of a class of coupled linear Reaction-Diffusion-Advection (RDA) Partial Differential Equations (PDEs) is tackled by exploiting the so-called "backstepping" approach (see Smyshlyaev et al. (2008); Krstic et al. (2004)).

Backstepping-based boundary controllers for several classes of scalar reaction-diffusion processes were presented, e.g., in Krstic et al. (2008); Liu (2003); Krstic et al. (2004). Scalar RDA processes are usually dealt with by resorting to a certain invertible transformation that removes the advection term from the PDE (see Smyshlyaev et al. (2008))), thus reducing the problem to that of stabilizing the resulting scalar reaction-diffusion PDE. As discussed in the Remark 1, this procedure can be applied to coupled RDA processes only when special restrictions on the corresponding parameters are in force, thus motivating the investigation of alternative solutions.

In recent years, the backstepping-based boundary stabilization of coupled PDEs is under intensive study (see Di Meglio et al. (2013); Vazquez et al. (2011); Di Meglio et al. (2012); Aamo (2013); Coron et al. (2011)) mostly referring to coupled hyperbolic processes of the transport-type. More recently, the boundary stabilization of linear coupled reaction-diffusion PDEs was addressed under the restriction that all the coupled processes possess the same diffusivity parameters (see Baccoli et al. (2014)) and in the general case in Baccoli et al. (2015a). These two distinct scenarios, indeed, imply deep differences in the solvability of the resulting kernel PDE.

The task of the present paper is to generalize the results presented in Baccoli et al. (2014) by including the advection term in the resulting system of coupled PDEs. The motivation to this investigation comes from chemical processes Orlov et al. (2002) where such equations occur to describe system dynamics, e.g., coupled temperature-concentration parabolic PDEs. As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to a kernel PDE with completely different boundary conditions than those obtained in Baccoli et al. (2014), and whose solution in explicit form cannot be found anymore. Additionally, preliminary investigations seem to suggest that when all processes possess their own diffusivity and advection coefficients the problem of boundary stabilization is unsolvable through the backstepping route. In this paper we therefore address the simplified case where all processes have the same diffusivity parameter, and we postpone to next investigation the more careful study of the general case.

The structure of the paper is as follows. Section II states the problem under investigation and introduces the underlying backstepping transformation. In Section III the solution of the kernel PDE is tackled, whereas in Section IV the proposed boundary control design and main stability result of this paper are drawn. Section V gives some concluding remarks and future perspectives of this research.

1.1 Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ and the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{1}$$

Throughout the paper we shall also utilize the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \texttt{ times}}, \tag{2}$$

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and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \tag{3}$$

for the corresponding norm of a generic vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in [L_2(0,1)]^n$. I_n denotes the identity matrix of dimension n.

2. PROBLEM FORMULATION AND BACKSTEPPING TRANSFORMATION

We consider a n-dimensional system of coupled reaction-advection-diffusion processes, equipped with Dirichlet-type boundary conditions, governed by the next vector-valued PDE

$$Q_t(x,t) = \theta Q_{xx}(x,t) + DQ_x(x,t) + \Lambda Q(x,t) \tag{4}$$

$$Q(0,t) = 0, \tag{5}$$

$$Q(1,t) = U(t) \tag{6}$$

where

$$Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n \tag{7}$$

is the vector collecting the state of all systems,

$$U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \Re^n \tag{8}$$

is the vector collecting all the manipulable boundary control signals, $\Lambda = \{\lambda_{ij}\} \in \Re^{n\times n}$ is the real-valued "reaction" matrix, $D \in \Re^{n\times n}$ is the diagonal "advection" matrix having the form $D = diag(d_i)$, with $d_i > 0\ \forall i = 1, 2, ..., n$, and $\theta \in \Re^+$ is a positive scalar. The open-loop system (4)-(6) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues when the matrix $\Lambda + \Lambda^T$ possesses sufficiently large positive eigenvalues.

Remark 1. Under the restriction

$$d_1 = d_2 = \ldots = d_n \equiv d, \tag{9}$$

the invertible change of variables

$$W(x,t) = Q(x,t)e^{\frac{d}{2\theta}x} \tag{10}$$

can be implemented which, after straightforward manipulations, analogous to those made in Smyshlyaev et al. (2008) to address the scalar case when $n = 1$, yields the advection-free transformed system of coupled PDEs

$$W_t(x,t) = \theta W_{xx}(x,t) + \left[\Lambda - \frac{d^2}{4\theta}I_n\right] W(x,t) \tag{11}$$

$$W(0,t) = 0, \tag{12}$$

$$W(1,t) = U(t)e^{\frac{d}{2\theta}} \tag{13}$$

whose stabilization can be addressed by following the procedure described in Baccoli et al. (2014). In the general case where the condition (9) is not fulfilled, such an approach is not feasible and another solution has to be found, which is the main goal of the present paper.

Here, we exploit the invertible backstepping transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \tag{14}$$

where $K(x,y)$ is a $n \times n$ matrix function whose elements are denoted as $k_{ij}(x,y)$ $(i,j = 1,2,\ldots,n)$ to exponentially stabilize system (4)-(6) by transforming it into the target system

$$Z_t(x,t) = \theta Z_{xx}(x,t) + DZ_x(x,t) - CZ(x,t) \tag{15}$$

$$Z(0,t) = 0, \tag{16}$$

$$Z(1,t) = 0, \tag{17}$$

where $Z(x,t) = [z_1(x,t),\ z_2(x,t), \ldots, z_n(x,t)]^T \in [L_2(0,1)]^n$ is the corresponding state vector and $C = \{c_{ij}\} \in \Re^{n,n}$ is an arbitrarily chosen real-valued square matrix.

The exponential stability properties of the target system (15)-(17), whose convergence rate can be made arbitrarily fast by a suitable choice of the matrix C, are investigated later in Theorem 2.

Following the usual backstepping design, we now derive and solve the PDE governing the kernel matrix function $K(x,y)$. Spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ take the form (the Leibnitz differentiation rule is used):

$$Z_x(x,t) = Q_x(x,t) - K(x,x)Q(x,t) - \int_0^x K_x(x,y)Q(y,t)dy \tag{18}$$

$$\begin{aligned} Z_{xx}(x,t) = {} & Q_{xx}(x,t) - \left[\frac{d}{dx}K(x,x)\right] Q(x,t) \\ & - K(x,x)Q_x(x,t) - K_x(x,x)Q(x,t) \\ & - \int_0^x K_{xx}(x,y)Q(y,t)dy \end{aligned} \tag{19}$$

where

$$\frac{d}{dx}K(x,x) = K_x(x,x) + K_y(x,x)$$

$$K_x(x,x) = \left.\frac{\partial K(x,y)}{\partial x}\right|_{y=x}, \quad K_y(x,x) = \left.\frac{\partial K(x,y)}{\partial y}\right|_{y=x}. \tag{20}$$

Using (4), and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is obtained in the form

$$\begin{aligned} Z_t(x,t) = {} & Q_t(x,t) - \int_0^x K(x,y)Q_t(y,t)dy \\ = {} & \theta Q_{xx}(x,t) + DQ_x(x,t) + \Lambda Q(x,t) \\ & - K(x,x)\theta Q_x(x,t) + K(x,0)\theta Q_x(0,t) \\ & + K_y(x,x)\theta Q(x,t) - K_y(x,0)\theta Q(0,t) \\ & - \int_0^x K_{yy}(x,y)\theta Q(y,t)dy - K(x,x)DQ(x,t) \\ & + K(x,0)DQ(0,t) + \int_0^x K_y(x,y)DQ(y,t)dy \\ & - \int_0^x K(x,y)\Lambda Q(y,t)dy. \end{aligned} \tag{21}$$

Combining (14), (19), (21) and performing lengthy but straightforward computations, yield