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Brief paper

An SOR implicit iterative algorithm for coupled Lyapunov equations[☆]

Ai-Guo Wu, Hui-Jie Sun*, Ying Zhang*

Harbin Institute of Technology, Shenzhen, Shenzhen 518055, PR China

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ABSTRACT

A novel implicit iterative algorithm is presented via successive over relaxation (SOR) iterations in this paper for solving the coupled Lyapunov matrix equation related to continuous-time Markovian jump linear systems. This algorithm contains a relaxation parameter, which can be appropriately chosen to improve the convergence performance of the algorithm. It has been shown that the sequence generated by the proposed algorithm with zero initial conditions monotonically converges to the unique positive definite solution of the considered equation. Moreover, some convergence results of the presented SOR implicit iterative algorithm with arbitrary initial conditions are established, and a method to choose the optimal relaxation parameter for this algorithm is given. Finally, two examples are provided to illustrate the effectiveness of the proposed algorithm.

1. Introduction

Coupled Lyapunov matrix equations (CLMEs) appear in the stability analysis of continuous-time Markovian jump linear systems. In Mariton (1988), the moment stability was discussed by the existence of unique positive definite solutions of the CLMEs. A necessary and sufficient condition was given by Ji and Chizeck (1990) for the stochastic stability of continuous-time Markovian jump systems in terms of the corresponding CLMEs. It was shown in Feng, Loparo, Ji, and Chizeck (1992) that the stochastic stability of jump linear systems can be characterized by the existence of unique positive definite solutions of the CLMEs.

Due to the aforementioned facts, the CLMEs have received considerable attention, and many effective algorithms have been proposed to solve them. In Jodar and Mariton (1987), the CLMEs were transformed into matrix–vector linear equations by using the Kronecker product, and its solutions can be directly obtained. However, this method suffers from high dimensionality. In Li, Zhou, Lam, and Wang (2011), some iterative algorithms were presented to solve the CLMEs associated with Itô Markovian jump stochastic systems. A parallel iterative algorithm was given by Borno (1995) to approximate the solutions of the continuous CLMEs. By applying the latest estimation some implicit iterative algorithms were developed by Wu and Duan (2015) to solve the discrete CLMEs and by Qian and Pang (2015) for solving the continuous CLMEs. The algorithm in Qian and Pang (2015) is a sequential version of the algorithm in Borno (1995). This is similar to Gauss–Seidel iterations vs Jacobi iterations for ordinary linear equations (Trefethen & Bau, 1997). Recently, by introducing some tunable parameters, an implicit iterative algorithm was developed by Wu, Duan, and Liu (2016) for solving the continuous CLMEs. It is known that the successive over relaxation (SOR) technique is a classical method for improving the convergence rate of the Jacobi iterations (Young, 2014). The SOR technique was used by Wu, Zhang, and Zhang (2018) to solve the discrete periodic Lyapunov matrix equations.

Inspired by the above facts, in this paper we aim to establish a novel iterative algorithm for solving the CLMEs related to continuous-time Markovian jump linear systems by using the idea of the SOR technique for the ordinary linear equations. First, the matrix equation is transformed into an equivalent form by introducing a relaxation parameter. Then, based on the transformed form a novel iterative algorithm, the SOR implicit iterative algorithm, is constructed for solving this kind of coupled Lyapunov matrix equations. Different from the algorithm in Qian and Pang (2015), a relaxation parameter is introduced in the proposed algorithm. This parameter can be chosen such that the algorithm achieves better convergence performance. Some convergence results of the presented SOR implicit iterative algorithm are established. In addition, an explicit expression is derived for the optimal relaxation parameter such that the algorithm achieves the fastest convergence rate. In addition, an easier method is proposed to compute the optimal parameter for a special case.

Throughout this paper, for a matrix $A \in \mathbb{R}^{n\times n}$, A^{T}, $f_A(s)$, and $\rho(A)$ denote its transpose, characteristic polynomial and spectral radius, respectively, and $\sigma(A)$ denotes the set of all its eigenvalues. For two integers $a \leq b$, the notation $\mathbb{I}[a, b]$ is defined as $\mathbb{I}[a, b] =$

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* Corresponding authors.
E-mail addresses: ag.wu@163.com (A.-G. Wu), huijie_s@163.com (H.-J. Sun), zhangyinghit@126.com (Y. Zhang).

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$\{a, a+1, \ldots, b\}$. The vectorization of a matrix $A = [a_1\ a_2 \cdots a_n] \in \mathbb{R}^{m\times n}$ is defined as $\text{vec}(A) = [a_1^{\text{T}}\ a_2^{\text{T}} \cdots a_n^{\text{T}}]^{\text{T}}$. The notation $\otimes$ represents the Kronecker product of two matrices, and $\|\cdot\|_2$ and $\|\cdot\|_{\text{F}}$ refer to the spectral norm and the Frobenius norm, respectively. Moreover, for two symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$, we write $X < Y$ if $X - Y$ is negative definite. It should be mentioned that, the sum is defined as zero if the upper limit of the sum notation is less than the lower limit.

2. Preliminaries and previous results

Consider a continuous-time Markovian jump linear system described by

$$\dot{x}(t) = A_{\rho(t)}x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\rho(t)$ is a time homogeneous Markovian process that takes values in a finite discrete set $S = \{1, 2, \ldots, N\}$. For the Markovian jump linear system (1), the system matrices of N subsystems are $A_i, i \in \mathbb{I}[1, N]$. The dynamic of the probability distribution of the Markovian chain is determined by

$$\dot{\varphi} = \varphi\Pi, \tag{2}$$

where φ is an N-dimensional row vector of unconditional probabilities, and $\Pi = [\pi_{ij}]_{N\times N}$ is the transition rate matrix. This matrix has the properties that $\pi_{ij} \geq 0$ for $j \neq i$, and $\sum_{j=1}^{N}\pi_{ij} = 0$, $i \in \mathbb{I}[1, N]$.

It has been known that the stochastic stability of the Markovian jump linear system (1)–(2) can be characterized by the existence of the unique positive definite solutions of the associated CLMEs.

Proposition 1 (*Ji & Chizeck, 1990*). *The Markovian jump linear system (1)–(2) is stochastically stable if and only if there exist unique positive definite matrices $P_i, i \in \mathbb{I}[1, N]$, satisfying the following CLMEs:*

$$A_i^{\text{T}}P_i + P_iA_i + \sum_{j=1}^{N}\pi_{ij}P_j = -Q_i,\ i \in \mathbb{I}[1, N], \tag{3}$$

where $Q_i, i \in \mathbb{I}[1, N]$, are arbitrarily given positive definite matrices.

Similarly to the treatment in Borno (1995), the CLMEs (3) can be equivalently expressed as

$$\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i = -Q_i - \sum_{j=1}^{i-1}\pi_{ij}P_j - \sum_{j=i+1}^{N}\pi_{ij}P_j,\ i \in \mathbb{I}[1, N], \tag{4}$$

with

$$\mathcal{A}_i = A_i + 0.5\pi_{ii}I,\ i \in \mathbb{I}[1, N]. \tag{5}$$

By Proposition 1, the following result can be easily derived from (4).

Proposition 2. *For any $i \in \mathbb{I}[1, N]$, the matrix $\mathcal{A}_i$ in (5) is Hurwitz stable if the Markovian jump linear system (1)–(2) is stochastically stable.*

Based on (4), two implicit iterative algorithms have been proposed by Borno (1995) and (Qian & Pang, 2015) to solve the CLMEs (3).

Lemma 1 (*Borno, 1995*). *Given the stochastically stable Markovian jump linear system (1)–(2), and positive definite matrices $Q_i > 0$, $i \in \mathbb{I}[1, N]$, the unique solution $(P_1, P_2, \ldots, P_N)$ of the corresponding CLMEs (3) can be obtained by the following iterative algorithm with $P_i(0) = 0$, $i \in \mathbb{I}[1, N]$:*

$$\begin{aligned}&\mathcal{A}_i^{\text{T}}P_i(m+1) + P_i(m+1)\mathcal{A}_i \\ &= -\sum_{j=1}^{i-1}\pi_{ij}P_j(m) - \sum_{j=i+1}^{N}\pi_{ij}P_j(m) - Q_i,\ i \in \mathbb{I}[1, N].\end{aligned} \tag{6}$$

That is, $\lim_{m\to\infty}P_i(m) = P_i$, $i \in \mathbb{I}[1, N]$.

Lemma 2 (*Qian & Pang, 2015*). *If the Markovian jump linear system (1)–(2) is stochastically stable, then the sequence $(P_1(m), P_2(m), \ldots, P_N(m))$ generated by the following algorithm with zero initial conditions*

$$\begin{aligned}&\mathcal{A}_i^{\text{T}}P_i(m+1) + P_i(m+1)\mathcal{A}_i \\ &= -\sum_{j=1}^{i-1}\pi_{ij}P_j(m+1) - \sum_{j=i+1}^{N}\pi_{ij}P_j(m) - Q_i,\ i \in \mathbb{I}[1, N],\end{aligned} \tag{7}$$

monotonically converges to the unique positive definite solution of the CLMEs (3) with $Q_i > 0$, $i \in \mathbb{I}[1, N]$.

For the algorithms (6) and (7), at each iteration step one needs to solve N standard continuous Lyapunov matrix equations in the form of $A^{\text{T}}X + XA = -Q$. Therefore, these two algorithms are in the implicit form.

3. Main results

In this section, we aim to give a novel implicit iterative algorithm to solve the CLMEs (3) by using the idea of the successive over relaxation (SOR) technique. For this end, we begin with a very simple identity. For the matrices $\mathcal{A}_i$, $i \in \mathbb{I}[1, N]$, in (5) and a scalar γ, the following relations hold

$$\begin{aligned}&\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i \\ &= (1-\gamma)\left(\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i\right) + \gamma\left(\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i\right),\ i \in \mathbb{I}[1, N].\end{aligned} \tag{8}$$

In the preceding section, it is known that the CLMEs (3) can be written as (4). Substituting (4) into the first term on the right-hand side of (8), gives

$$\begin{aligned}&\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i \\ &= (1-\gamma)\left(-\sum_{j=1}^{i-1}\pi_{ij}P_j - \sum_{j=i+1}^{N}\pi_{ij}P_j - Q_i\right) \\ &\quad + \gamma\left(\mathcal{A}_i^{\text{T}}P_i + P_i\mathcal{A}_i\right),\ i \in \mathbb{I}[1, N].\end{aligned} \tag{9}$$

With the idea of using the latest updated estimation in Wu and Duan (2015) in mind, from the preceding relations in (9) the following implicit iterative algorithm can be constructed to solve the CLMEs (3):

$$\begin{aligned}&\mathcal{A}_i^{\text{T}}P_i(m+1) + P_i(m+1)\mathcal{A}_i \\ &= (1-\gamma)\left[-\sum_{j=1}^{i-1}\pi_{ij}P_j(m+1) - \sum_{j=i+1}^{N}\pi_{ij}P_j(m) - Q_i\right] \\ &\quad + \gamma[\mathcal{A}_i^{\text{T}}P_i(m) + P_i(m)\mathcal{A}_i],\ i \in \mathbb{I}[1, N],\end{aligned} \tag{10}$$

where $\mathcal{A}_i$, $i \in \mathbb{I}[1, N]$, are given in (5) and γ is the relaxation parameter.

Remark 1. If the parameter γ is chosen as $\gamma = 0$, then the proposed algorithm (10) is reduced to the algorithm (7).

Remark 2. Similarly to the algorithms (6) and (7), N standard continuous Lyapunov matrix equations need to be solved at each iteration step in the presented algorithm (10). Therefore, the algorithm (10) is also in an implicit form.

Remark 3. It could be seen that the relation of the proposed algorithm (10) with the algorithm in Qian and Pang (2015) is very similar to that of Gauss–Seidel iterations vs the SOR iterations in ordinary linear equations. For convenience, in this paper the proposed algorithm (10) is called the SOR implicit iterative algorithm.