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General linear forward and backward Stochastic difference equations with applications☆

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ABSTRACT

In this paper, we consider a class of general linear forward and , backward stochastic difference equations (FBSDEs) which are fully coupled. The necessary and sufficient conditions for the existence of a (unique) solution to FBSDEs are given in terms of a Riccati equation. Two kinds of stochastic LQ optimal control problem are then studied as applications. First, we derive the optimal solution to the classic stochastic LQ problem by applying the solution to the associated FBSDEs. Secondly, we study a new type of LQ problem governed by a forward–backward stochastic system (FBSS). By applying the maximum principle and the solution to FBSDEs, an explicit solution is given in terms of a Riccati equation. Finally, by exploring the asymptotic behavior of the Riccati equation, we derive an equivalent condition for the mean-square stabilizability of FBSS.

1. Introduction

Forward and backward stochastic difference equations (FBSDEs) have wide applications in modern engineering and applied mathematics (Fleming & Stein, 2004). They are a new type of stochastic difference equations (SDEs) which can be roughly treated as two-point boundary valued problems with a stochastic feature. One key component of FBSDEs is the backward stochastic difference equation (BSDE). Recall that the backward equation in the deterministic case is a recursive equation from the terminal time and its solution can be obtained by iterations. See Lewis, Vrabie, and Syrmos (2012) for its application in the linear quadratic optimal control. This, however, is no longer possible for the stochastic case where the solution of the BSDE must be adapted (Zhang, Li, Xu, & Fu, 2015). In mathematical description, a conditional expectation is involved in the equation, which makes it difficult to solve. One of the few types of such equations arises from solving the optimal control of Hamiltonian systems via the maximum principle in optimal control problems (Wonham, 1968). Bismut (1978) studied the optimal control via the stochastic Hamiltonian systems and the Pontryagin's maximum principle. When the FBSDEs are decoupled or partially coupled, the solution can be obtained by firstly solving the decoupled equation. For some specific type of FBSDEs such as stochastic Hamiltonian system, Zhang et al. (2015) characterized the solution by establishing a relationship between the forward and backward variables. Xu, Xie, and Zhang (2017) studied the solvability of a specific class of infinite-horizon FBSDEs and established an equivalence between the exponential stabilizability of the stochastic control system and the FBSDEs.

For the continuous-time case, there are three main methods for solving a fully coupled FBSDEs: the compressed mapping method, the four step scheme (Ma, Protter, & Yong, 1994), which provides an explicit relation between the forward and backward variables via a quasilinear partial differential equation and the method of continuation by Hu and Peng (1995). More results can be found in Antonelli (1993), Ma and Yong (1999), Pardoux and Peng (1990), Peng and Shi (2000), Peng and Wu (1999), Tang (2013), Wu (2005, 2013), Yong (1997, 1999) and the references therein. Moreover, some decoupled numerical schemes were proposed for coupled forward and backward stochastic differential equations in Zhao, Fu, and Zhou (2014) and references therein.

Control of stochastic systems governed by FBSDEs is of great significance in stochastic control theory and stochastic game theory (Yong, 2006). A general maximum principle was proposed for the optimal control of a forward–backward stochastic system (FBSS) in Wu (2013). The maximum principle for the FBSS with terminal state constraints was given in Ji and Wei (2013). Wang, Wu, and Xiong (2015) studied the LQ control problem for FBSS with partial information. More results about the continuous-time

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case can be found in Huang, Li, and Shi (2012) and the references therein. Though much work has been done for the continuous-time case, there is little progress for the solvability of general discrete-time FBSDEs and the corresponding control problems. The main challenge is due to the involvement of the conditional expectations and the coupling between the forward and backward processes.

In this paper, we will study a class of general fully coupled linear FBSDEs. Firstly, by establishing a homogeneous relationship between the forward and backward processes, we derive the necessary and sufficient conditions for the (unique) solvability of the FBSDEs. Secondly, we consider two kinds of LQ optimal control problems. On one hand, the solution of the FBSDEs is applied to solve the classic LQ optimal control problems which leads to the same result as in the literature. On the other hand, we study a new type of LQ control problem governed by the FBSS whose optimal solution is characterized by FBSDEs in terms of the maximum principle. By applying the solution to FBSDEs, the explicit optimal solution is given in terms of the solution to a Riccati equation. Finally, by exploring the limit property of the Riccati equation, it is shown that the FBSS is mean-square exponentially stabilizable if and only if an algebraic Riccati equation has a particular solution.

The remainder of the paper is organized as follows. Section 2 discusses the solvability of the FBSDEs. The applications to the optimal control problems are presented in Section 3. Numerical examples are illustrated in Section 4. Some concluding remarks are given in Section 5. The proofs of main results are given in the Appendix.

The following notations will be used throughout this paper: R^n denotes the family of n dimensional vectors. $R^{n\times m}$ denotes the family of $n\times m$ dimensional matrices. x' means the transpose of x. $M>0(\geq 0)$ means that M is symmetric and positive definite (positive semi-definite). $N^{\dagger}$ represents the Moore–Penrose inverse of a matrix N. $Range(\cdot)$ denotes the range. $(\Omega,\mathcal{F},\mathcal{P},\{\mathcal{F}_k\}_{k\geq 0})$ denotes a complete probability space on which a scalar white noise w_k is defined such that $\{\mathcal{F}_k\}_{k\geq 0}$ is the natural filtration generated by w_k, i.e., $\mathcal{F}_k=\sigma\{w_0,\ldots,w_k\}$, and augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. $E[x_k|\mathcal{F}_s]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_s$. Denote also $E_k(\cdot)=E[\cdot|\mathcal{F}_k]$. Define the inner product $\langle x,y\rangle = x'y$. We finally introduce an admissible set: $\mathcal{U}[0,N]=\{u_k, k=0,\ldots,N|u_k \text{ is } \mathcal{F}_{k-1} \text{ adapted}, E\sum_{k=0}^{N}\|u_k\|^2<\infty\}$.

2. Forward and backward stochastic difference equations

In this paper, we mainly consider the FBSDEs given by:

$$\begin{cases} x_{k+1}=A_kx_k+B_kE_{k-1}(C_k'\lambda_k),\ x_l=\xi,\\ \lambda_{k-1}=E_{k-1}[D_k\lambda_k]+Qx_k,\ \lambda_N=Hx_{N+1}, \end{cases} \tag{1}$$

where $x_k\in R^n$ is the forward component, and $\lambda_k\in R^m$ is the backward component. $A_k=A+\bar{A}w_k$, $B_k=B+\bar{B}w_k$, $C_k=C+\bar{C}w_k$, $D_k=D+\bar{D}w_k$ with $A,\bar{A}\in R^{n\times n}$, $B,\bar{B}\in R^{n\times r}$, $C,\bar{C}\in R^{m\times r}$, $D,\bar{D}\in R^{m\times m}$ are constant matrices and w_k is a white noise with zero mean and unit covariance. $Q,H\in R^{m\times n}$ are constant matrices. The initial value $\xi\in R^n$ is a constant vector. l and N are integers with $0\leq l\leq N$.

It is noticed that linear FBSDEs (1) is of a general form and is fully coupled. The solvability of (1) has wide applications in the stochastic optimal control problem of discrete-time systems with multiplicative noise. Taking the stochastic LQ optimal control problem for example, the optimal controller is established by fully coupled FBSDEs which are a special case of (1). Moreover, the first-order discrete approximation (Boucharda and Touzi, 2004; Gobeta and Labart, 2007; Zhang, 2004) of the forward and backward stochastic differential equations has a similar form of FBSDEs (1). In fact, consider the following forward and backward stochastic differential equations:

$$dx(t)=[A_1x(t)+B_1p(t)+B_2q(t)]dt+[\bar{A}_1x(t)$$
$$+\bar{B}_1p(t)+\bar{B}_2q(t)]dw(t), \tag{2}$$
$$dp(t)=-[D_1p(t)+\bar{D}_1q(t)+\bar{Q}x(t)]dt+q(t)dw(t), \tag{3}$$

where all the vectors and the matrices have appropriate dimensions. Given a uniform partition: $0=t_0<t_1<\cdots<t_n=T$, let $\delta=t_{k+1}-t_k$, $\Delta w_k=w_{k+1}-w_k$, $\mathcal{F}_k=\sigma\{\Delta w_0,\ldots,\Delta w_k\}$. Simply denote the simulation variables at time t_k as x_k, p_k and q_k. Denote $q_k=\frac{1}{\delta}E_{k-1}[\Delta w_kp_k]=\frac{1}{\delta}E[\Delta w_kp_k|\mathcal{F}_{k-1}]$, we have the discretization of (2)–(3) given by

$$\begin{aligned} x_{k+1}&=(I+\delta A_1+\bar{A}_1\Delta w_k)x_k+(\delta B_1+\bar{B}_1\Delta w_k)\\ &\quad\times E_{k-1}(p_k)+\frac{1}{\delta}(\delta B_2+\bar{B}_2\Delta w_k)\\ &\quad\times E_{k-1}(\Delta w_kp_k), \end{aligned}\tag{4}$$

$$p_{k-1}=E_{k-1}[(I+\delta D_1+\Delta w_k\bar{D}_1)p_k]+\delta\bar{Q}x_k. \tag{5}$$

It follows that the numerical simulation (4)–(5) has a similar structure to the FBSDEs (1). In this sense, the solvability of (1) will open up a new way to solve the fully coupled forward and backward stochastic differential equations. These significant applications motivate us to study the solvability of FBSDEs (1). The main difficulties lie in that the forward and backward components are fully coupled and the conditional expectation is involved. Let us start with some definitions of the solvability to FBSDEs (1).

Definition 1.

(1) A pair (x_k,λ_{k-1}) is called an adapted solution of FBSDEs (1) associated with (l,ξ) if it satisfies (1) for $k=l,\ldots,N$, that is, the following holds for $k=l,\ldots,N$, almost surely,

$$x_{k+1}=\Phi_1(k,l)x_0+\sum_{i=l}^{k}\Phi_1(k,i+1)B_iE_{i-1}(C_i'\lambda_i),$$

$$\lambda_{k-1}=E_{k-1}[\Phi_2(k,N)\lambda_N]+\sum_{i=k-1}^{N-1}E_{k-1}[\Phi_2(k,i)$$
$$\times Qx_{i+1}],$$

where $\Phi_1(k,i)=A_k\cdots A_i$, $i\leq k$, $\Phi_1(k,i)=I$, $i>k$ and $\Phi_2(k,i)=D_k\cdots D_i$, $i\geq k$, $\Phi_2(k,i)=I$, $i<k$.

(2) When (1) admits a (unique) solution associated with (l,ξ) where $l=0,\ldots,N$, $\xi\in R^n$, we say that (1) is (uniquely) solvable associated with (l,ξ). FBSDEs (1) are said to be (uniquely) solvable at l if they are (uniquely) solvable for all $\xi\in R^n$. FBSDEs (1) are said to be (uniquely) solvable if they are (uniquely) solvable at all $l=0,\ldots,N$.

For convenience of the future use, we introduce the following general Riccati equation:

$$P_k=\begin{bmatrix}DP_{k+1} & \bar{D}P_{k+1}\end{bmatrix}\Upsilon_{k+1}^{\dagger}\begin{bmatrix}A\\ \bar{A}\end{bmatrix}+Q,\ P_{N+1}=H, \tag{6}$$

where

$$\Upsilon_{k+1}=I-\begin{bmatrix}BC'P_{k+1} & B\bar{C}'P_{k+1}\\ \bar{B}C'P_{k+1} & \bar{B}\bar{C}'P_{k+1}\end{bmatrix}. \tag{7}$$

Firstly, we study the solvability of FBSDEs (1) associated with (l,ξ). An equivalent condition is given in terms of a range condition associated with the solution of a state equation with the initial value ξ.

2.1. Solvability of FBSDEs associated with (l,ξ)

Theorem 1. *FBSDEs (1) is solvable associated with (l,ξ) if and only if*

$$\begin{bmatrix}A\\ \bar{A}\end{bmatrix}x_k\in Range(\Upsilon_{k+1}),\ k=l,\ldots,N, \tag{8}$$