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Unified stability criteria for slowly time-varying and switched linear systems☆

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ABSTRACT

This paper presents a unified approach to formulating stability conditions for slowly time-varying linear systems and switched linear systems. The concept of total variation is generalized to the case of matrix-valued functions. Using this generalized concept, a result extending existing stability conditions for slowly time-varying linear systems is derived. As special cases of this result, two sets of stability conditions are derived for switched linear systems, which match known results in the literature. A numerical example is included to further illustrate the application of the main result.

1. Introduction

Stability of slowly time-varying linear systems and switched linear systems has been extensively studied during the past decades. Earlier results on stability of slowly time-varying linear systems were derived via the frozen-time approach (Amato, Celentano, & Garofalo, 1993; Coppel, 1978; Desoer, 1969; Ilchmann, Owens, & Prätzel-Wolters, 1987; Ioannou & Sun, 1996). Specifically, if the system is stable for any frozen time and varies slowly enough, then the system is globally exponentially stable. There are two main ways in the literature to characterize the rate of system variation. First, in the work of Amato et al. (1993), Coppel (1978) and Desoer (1969), it is shown that the system is globally exponentially stable if the time derivative of the system matrix is sufficiently small. Second, in the work of Ilchmann et al. (1987) and Ioannou and Sun (1996), global exponential stability is established under either of the following two conditions: (1) the system matrix is globally Lipschitz in time and the Lipschitz constant is sufficiently small; (2) the time integral of the norm of the time derivative of the system matrix is bounded by some affine function of the length of the time interval, and the slope of the affine function is sufficiently small.

The above earlier results all impose conditions on the stability of the system matrix at each instant of time as well as on the continuity of the system matrix, which are somewhat conservative. More recent works on stability of linear time-varying systems have relaxed these conditions (Jetto & Orsini, 2009; Solo, 1994; Zhang, 1993). In these works, stability conditions were derived via a different approach, which was based on the "perturbed frozen-time form" of linear time-varying systems (Jetto & Orsini, 2009). In particular, it was assumed that the system matrix is stable at an infinite sequence of times. Then, the linear time-varying system can be viewed as a combination of a switched linear system with stable subsystems and a perturbation term. It was shown that the linear time-varying system is globally exponentially stable if, for each time interval between two consecutive stable times, the length of the time interval is long enough and the perturbation of the system matrix over the time interval is small enough.

For a switched linear system, the results by Morse (1996), as well as by Hespanha and Morse (1999), stated that if each subsystem is stable and if the system switches sufficiently slowly, then the switched system is also stable. The rate of switching is characterized by the *dwell-time*, or the *average dwell-time*, which describes the time, or average time, respectively, between two successive switches. One extension by Zhai et al. Zhai, Hu, Yasuda, and Michel (2001) relaxed the assumption on the stability of all subsystems, by allowing switched linear systems with unstable subsystems. It was shown that the system is exponentially stable if the average dwell-time is sufficiently large and the ratio between the activation time of unstable subsystems and the activation time

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of stable subsystems is sufficiently small. Following this line of extension, a stabilization result was derived in Zhao, Yin, Li, and Niu (2015) for switched linear systems with only unstable subsystems. Another branch of extensions is to allow different dwell-times or average dwell-times for each subsystem or each pair of subsystems. The former refers to the time (see Blanchini, Casagrande, & Miani, 2010), or the average time (see Zhao, Zhang, Shi, & Liu, 2012), that each subsystem is activated before the system switches to other subsystems. It was shown that the switched system is stable if the dwell-time or average dwell-time for each subsystem is large enough. The latter refers to the elapsed time (see Blanchini et al., 2010; Langerak & Polderman, 2005), or the average elapsed time (see Kundu & Chatterjee, 2015), before the system switches from one subsystem to another in each transition pair. It was shown that the switched system is stable if dwell-time or average dwell-time for each transition pair is large enough. (Other stability results in terms of constrained transition pairs governed by a graph can be found in Lee & Dullerud, 2007; Philippe, Essick, Dullerud, & Jungers, 2016 and the references therein.)

It is natural to view switched linear systems as a special class of linear time-varying systems. Although there are some similarities, to the best of our knowledge, there is no explicit relationship bridging the two sets of stability results. To be more specific, the stability conditions available in one set cannot be applied directly to the other.[1] With this in mind, we study in this paper the gap between the two sets of results. Our aim is to build connections between stability results for slowly time-varying linear systems derived by the frozen-time approach and stability results for switched linear systems with stable subsystems. Inspired by the total variation of vector-valued functions, we first extend the concept of total variation to matrix-valued functions (Section 2). Using this extended concept, we derive a generalized stability result for slowly time-varying linear systems (Section 2), where the system matrix could be piecewise differentiable, with discontinuities at the non-differentiable points. Then, we apply the derived result to switched linear systems, and obtain two sets of stability conditions (Section 3), which we show to match the existing results in Hespanha and Morse (1999) and Kundu and Chatterjee (2015). Next, we present a numerical example to further illustrate how our main results can be applied to establish stability of slowly time-varying linear systems (Section 4). Finally, we draw conclusions and discuss several future directions (Section 5).

Notation: We denote by $\|\cdot\|$ the Euclidean norm of a vector and the induced norm for a matrix. We write A^T for the transpose of a matrix A. We use I to denote the $n \times n$ identity matrix. For any complex number a, we use $\mathrm{Re}\{a\}$ to denote the real part of a. For an $n \times n$ matrix A, we use $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ to denote the eigenvalues of A. The matrix A is called Hurwitz if $\mathrm{Re}\{\lambda_i(A)\} < 0$, for all $i \in \{1, 2, \ldots, n\}$. A continuous function $\alpha(\cdot) : [0, \infty) \to [0, \infty)$ is said to be a class $\mathcal{K}_\infty$ function if $\alpha(0) = 0$, $\alpha(\cdot)$ is strictly monotone increasing, and $\alpha(\cdot)$ is unbounded.

2. Stability of slowly time-varying linear systems

Consider a real n-dimensional linear time-varying system described by

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

where $t \in [0, \infty)$ is the time, $x(\cdot) \in \mathbb{R}^n$ is the state vector, and $A(\cdot) \in \mathbb{R}^{n\times n}$ is the system matrix. The system described by (1) is said to be *globally exponentially stable* if there exist finite positive constants C and ρ such that for any initial condition $x(0) \in \mathbb{R}^n$, the state vector $x(\cdot)$ satisfies

$$\|x(t)\| \le C\|x(0)\|e^{-\rho t} \quad \forall\, t \ge 0.$$

In the special case when $A(t) = A \;\; \forall\; t \ge 0$, it is well known that the system (1) is globally exponentially stable if and only if A is Hurwitz. The conditions for global exponential stability of (1) have been widely studied (Amato et al., 1993; Coppel, 1978; Desoer, 1969; Ilchmann et al., 1987; Ioannou & Sun, 1996). Here, we present two existing results for slowly time-varying systems. We first introduce an assumption on the boundedness and Hurwitzness of the system matrix.

Assumption 1. The system matrix $A(\cdot)$ is such that

(i) There exists $L > 0$ such that $\|A(t)\| \le L \;\; \forall\, t \ge 0$.
(ii) There exists $\kappa > 0$ such that $\mathrm{Re}\big\{\lambda_i\big(A(t)\big)\big\} \le -\kappa \quad \forall\, t \ge 0,\; i \in \{1, 2, \ldots, n\}$.

By the above two conditions, for any $\lambda \in (0, \kappa)$, there exists $c > 0$ (which depends on L, κ, and λ, see Khalil (2002, Section 9.6, proof of Lemma 9.9) such that

$$\|e^{A(t)s}\| \le ce^{-\lambda s} \quad \forall\, t \ge 0,\; s \ge 0. \tag{2}$$

In Section 4, we will provide an example demonstrating how c and λ can be computed in a specific case.

Theorem 1 (*Desoer, 1969*)**.** *The system (1) is globally exponentially stable if the following conditions are satisfied:*

(i) *$A(\cdot)$ satisfies Assumption 1.*
(ii) *$A(\cdot)$ is differentiable and $\|\dot{A}(t)\| \le \frac{4\lambda^2}{3c^4} \;\; \forall\, t \ge 0$, where c and λ are from (2).*

Here $\|\dot{A}(\cdot)\|$ can be regarded as the rate at which the system changes over time. Hence, the result of Theorem 1 implies that a linear time-varying system (1) is globally exponentially stable if the system matrix is Hurwitz for each fixed time, uniformly bounded, and changes at a sufficiently small rate.

A more general sufficient condition is obtained by replacing $\|\dot{A}(\cdot)\|$ with the integral of $\|\dot{A}(\cdot)\|$ over a time interval, as follows.

Theorem 2 (*Theorem 3.4.11 in Ioannou & Sun, 1996*)**.** *The system (1) is globally exponentially stable if the following conditions are satisfied:*

(i) *$A(\cdot)$ satisfies Assumption 1.*
(ii) *$A(\cdot)$ is differentiable and there exist scalars $\alpha > 0$ and $0 < \mu < \frac{\beta_1}{2\beta_2^3}$ such that*

$$\int_t^{t+T} \big\|\dot{A}(s)\big\| ds \le \mu T + \alpha \quad \forall\, t \ge 0,\; T \ge 0,$$

where $\beta_1 = \frac{1}{2L}$, $\beta_2 = \frac{c^2}{2\lambda}$, and L, c, λ are from Assumption 1 and (2).

The third condition is in terms of the integral of $\|\dot{A}(\cdot)\|$ on each interval $[t, t+T]$, which is required to be bounded by some affine function of the length of the time interval and the slope of the affine function is sufficiently small. The third condition is also called "nondestabilizing condition" in Morse (1990).

All the sufficient conditions above assume that $A(\cdot)$ is differentiable over $[0, \infty)$. In the sequel, we will relax this assumption and consider a more general case in which $A(\cdot)$ is only piecewise differentiable. Our approach will entail appealing to total variation of piecewise differentiable functions.

[1] In Jetto and Orsini (2009), it was mentioned that the derived results can be applied to switched linear systems. It can be shown by applying the results in Jetto and Orsini (2009) that a switched linear system is globally exponentially stable if each subsystem is stable and the dwell time is large enough. However, it seems difficult to use this method to derive more advanced results on switched linear systems discussed in Section 3.2.