Automatica 96 (2018) 380–392

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Minimal controllability time for finite-dimensional control systems under state constraints ☆

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ARTICLE INFO

Article history:
Received 16 February 2018
Received in revised form 3 June 2018
Accepted 15 June 2018

Keywords:
Linear control systems
State constraints
Minimal time
Brunovsky normal form

ABSTRACT

We consider the controllability problem for finite-dimensional linear autonomous control systems, under state constraints but without imposing any control constraint. It is well known that, under the classical Kalman condition, in the absence of constraints on the state and the control, one can drive the system from any initial state to any final one in an arbitrarily small time. Furthermore, it is also well known that there is a positive minimal time in the presence of compact control constraints. We prove that, surprisingly, a positive minimal time may be required as well under state constraints, even if one does not impose any restriction on the control. This may even occur when the state constraints are unilateral, like the nonnegativity of some components of the state, for instance. Using the Brunovsky normal forms of controllable systems, we analyze this phenomenon in detail, that we illustrate by several examples. We discuss some extensions to nonlinear control systems and formulate some challenging open problems.

1. Introduction

Let $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$ be integers, with $m < n$. Let A be an $n\times n$ matrix and B be an $n \times m$ matrix, with real coefficients, satisfying the Kalman condition

$$\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n. \tag{1}$$

Throughout the paper, we consider the linear autonomous control system

$$\dot{y}(t) = Ay(t) + Bu(t) \tag{2a}$$

with some initial condition

$$y(0) = y^0 \tag{2b}$$

where $y(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control. In order to avoid confusion, we will sometimes write $y(t; u)$ the solution of (2). It is well known that, given any two points y^0 and y^1 of $\mathbb{R}^n$ and given any time $T > 0$, there exists a control $u \in L^\infty((0,T),\mathbb{R}^m)$ such that the corresponding trajectory, solution of (2), satisfies $y(T) = y^1$. In other words, one can pass from any initial condition to any final one *in arbitrarily small time*. Of course, this is at the price of using controls that have a L^∞-norm that is larger as the transfer time T is smaller (Seidman & Yong, 1996). Therefore, under the Kalman condition (1), if there is no state and control constraint in the control problem, then the *minimal controllability time*, defined as the infimum of times required to pass from y^0 to y^1, is equal to 0.

Now, we consider a connected subset $C \subset \mathbb{R}^n$ with nonempty interior, standing for state constraints that we want to impose to

☆ The second author acknowledges the support of the ANR project Finite 4SoS ANR-15-CE23-0007-01. The work of the third author was funded by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement no. 694126-DyCon), the Grants MTM2014-52347 and MTM2017-92996 of MINECO (Spain), the Grant FA9550-18-1-0242 of AFOSR and the Grant ICON of the French ANR. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Denis Arzelier under the direction of Editor Richard Middleton.

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https://doi.org/10.1016/j.automatica.2018.07.010
0005-1098/

the controllability problem, and we address the following question:

Given any two points y^0 and y^1 in C, is it possible to steer the control system (2) from y^0 to y^1 in arbitrarily small time T, while guaranteeing that $y(t) \in C$ for every $t \in [0, T]$, or is there a positive minimal time required?

We stress that we impose *no control constraint*, i.e., $u(t) \in \mathbb{R}^m$, but we impose a state constraint. It is surprising that this apparently simple question has not been investigated before. It is the main objective of this paper to explore it.

Before going further, it is useful to note that controllability under control constraints but without state constraints is well understood (see Brammer, 1972) and can be studied by usual optimal control methods (see Lee & Markus, 1967; Trélat, 2005). Recall that, when there is a control constraint $u(t) \in \Omega$ with Ω a compact subset of $\mathbb{R}^m$ then the set $\mathrm{Acc}_\Omega(y^0, T)$ of accessible points from y^0 in time $T > 0$ with controls $u \in L^\infty((0, T), \Omega)$ is compact and convex and evolves continuously with respect to T: hence in this case the minimal time required to pass from y^0 to $y^1 \neq y^0$ is always positive.

Here, in contrast, we want to investigate the question of knowing whether the minimal time may be positive when imposing state constraints but no control constraint. Of course, in order to address this question we may first wonder whether the target point y^1 is reachable or not from the initial point y^0. This is the question of controllability under state constraints, which is not the objective of the present paper but on which we shortly comment hereafter.

Controllability under state constraints. When C is a proper subset of $\mathbb{R}^n$, the question of controllability under the state constraint $y(\cdot) \in C$ is complicated. Even under the Kalman condition, very simple state constraints can immediately make fail the controllability property. For instance, take the control system in $\mathbb{R}^2$

$$\dot y_1(t) = y_2(t), \quad \dot y_2(t) = u(t),$$

and the state constraint $y_2(t) \geqslant 0$ on the second component of the state. For any trajectory, the first component $y_1(t)$ must be nondecreasing, and then obviously one cannot pass from any point to any other.

Controllability under state constraints has not been much investigated in the literature, certainly due to the difficulty of the question, even for linear control systems. Early conditions were given in Krastanov and Veliov (1992), with the idea of deriving conic directions of expansion of the reachable set. A more achieved version appears in Krastanov (2008), where the author states a necessary and sufficient condition for small-time controllability of linear control systems under conic state constraints. The verification of such algebraic conditions (given in terms of convex hull) remain however quite technical. In Heemels and Camlibel (2007), controllability is established under appropriate invertibility conditions of the transfer matrix and adequate Hautus test conditions. We also mention the recent paper (Le & Marigonda, 2017) for sufficient controllability conditions for nonlinear control systems.

In this paper, our objective is, when we already know that y^1 can be reached from y^0 under state constraints, to investigate whether the minimal time may be positive or not, while keeping the connecting trajectory in the set C.

It is anyway interesting to note that there is a specific situation under which controllability under state constraints can easily be proved, within a transfer time that may however be quite large. This is when y^0 and y^1 are *steady-states* (a point $\bar y \in C$ is a *steady-state* if there exists $\bar u \in \mathbb{R}^m$ such that $A\bar y + B\bar u = 0$). This situation is studied in Section 4.1. More precisely, we prove in this section that, under a slight condition on the set C (which is satisfied if C is convex), it is possible to steer the control system (2) from any steady-state $y^0 \in \mathring{C}$ to any steady-state $y^1 \in \mathring{C}$ in time sufficiently large, while ensuring that the corresponding trajectory remains in the interior $\mathring{C}$ of the set C. The proof is done by an iterative use of a local controllability result along a path of steady-states (whence the possibly large transfer time). The question is then to know whether one could find a control for which this transfer time would be arbitrarily small.

Minimal controllability time. We investigate the minimal time problem for the system (2) under state constraints $y(\cdot) \in C$, without control constraint. Given $y^0, y^1 \in C$, we define $T_C(y^0, y^1)$ as the infimum of times required to pass from y^0 to y^1 under the state constraint C (with an unconstrained L^∞ control), with the agreement that $T_C(y^0, y^1) = +\infty$ if y^1 is not reachable from y^0. More precisely, if y^1 is reachable from y^0, with L^∞ controls, we define

$$\begin{aligned} T_C(y^0, y^1) = \inf\big\{T > 0 \ \mid\ &\exists u \in L^\infty((0, T), \mathbb{R}^m) \text{ s.t.} \\ &\text{the solution } y(t) \text{ of (2) satisfies} \\ &\quad y(T) = y^1 \text{ and } \forall t \in [0, T],\ y(t) \in C\big\}. \end{aligned}$$

It is obvious that if $r = \operatorname{rank} B = n$ then $T_C(y^0, y^1) = 0$ for any y^0 and y^1 belonging to a same connected component of C. More precisely, given any time $T > 0$ and any C^1-path, $\bar y$ such that $\bar y(0) = y^0, \bar y(T) = y^1$ and $\bar y(t) \in C$ for every $t \in [0, T]$, then any control $\bar u$, satisfying $B\bar u(t) = \dot{\bar y}(t) - A\bar y(t)$, steers the solution of (2) to y^1 in time T. Hence, in what follows we assume that $r < n$.

As a first example (more details are given in Example 10 further), consider the linear control system

$$\dot y_1(t) = y_1(t) + u(t), \qquad \dot y_2(t) = 2y_2(t) + u(t),$$

under the nonnegativity state constraints $y_1(\cdot) \geqslant 0, y_2(\cdot) \geqslant 0$, and take the terminal conditions $y^0 = (1, 1/2)^\top$ and $y^1 = (2, 1)^\top$. Both points are steady-states, $C = [0, +\infty)^2$ is convex and the Kalman rank condition is satisfied. Hence, it is possible to steer the system from y^0 to y^1 with a trajectory satisfying the state constraints (see Section 4.1). But we claim that this cannot be done in arbitrarily small time. Here, the value of the minimal time under the state constraint $y(\cdot) \in C$ is $T_C(y^0, y^1) = \ln(2)$. In contrast, steering the control system from y^1 to y^0 in C can be done in arbitrarily small time, i.e., we have $T_C(y^1, y^0) = 0$.

The main result of the paper is the following.

Theorem 1. *Let C be a subset of IR^n and let $y^0 \in C$.*

(i) *Let $y^1 \in \mathring{C} \setminus \{y^0\}$. Assume that $y^0 \in \mathring{C}$ and there exists a steady state $\bar y \in \mathring{C}$. If $\bar y$ and y^1 are in a same connected component of $\big(\{y^0\} + \mathrm{Ran}\, B\big) \cap \mathring{C}$, then $T_C(y^0, y^1) = 0$.*

(ii) *Assume that C is bounded and let $y^1 \in C \setminus \{y^0\}$. If $y^1 - y^0 \notin \mathrm{Ran}(B)$ then $T_C(y^0, y^1) > 0$.*

(iii) *Assume that*

$$C = \{y \in IR^n \ \mid\ \langle \ell, y\rangle = \ell_1 y_1 + \cdots + \ell_n y_n \geqslant \beta\} \tag{3}$$

(unilateral and affine state constraint) for some $\beta \in IR$ and for some generic[1] $\ell \in IR^n \setminus \{0\}$.

- *If $r = \operatorname{rank} B > 1$ then under a generic[2] condition on the pair (A, B) we have $T_C(y^0, y^1) = 0$ for any $y^0, y^1 \in \mathring{C}$.*
- *If $r = 1$ or if the above generic condition is not satisfied then there exists an open subset $C_1 \subset C$ such that $T_C(y^0, y^1) > 0$ for every $y^1 \in C_1$.*

[1] The word generic means here that ℓ belongs to some subset of $\mathbb{R}^n \setminus \{0\}$ which is open and dense.

[2] The word generic means here that the pair (A, B) belongs to an open and dense subset of the set of matrices (A, B) with A (resp. B) an $n \times n$ (resp. $n \times m$) matrix.