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Reachability analysis of linear dynamic systems with constant, arbitrary, and Lipschitz continuous inputs ☆

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ABSTRACT

This paper sets forth a method for reachability analysis of linear dynamic systems in continuous time that can be used to compute time-domain bounds of states and outputs with floating-point precision. The focus is on the particular initial conditions and inputs that cause state or output trajectories to attain their extreme values in time. Inputs can be constant, arbitrary, or Lipschitz continuous waveforms. Uncertainties in initial conditions and inputs are modeled using zonotopes. The calculations exploit the definition of zonotopes and the modal information of the linear system dynamics.

1. Introduction

In the context of dynamic systems, reachability analysis refers to the computation of sets that contain all possible trajectories of the states originating from uncertain initial conditions and driven by uncertain inputs (Althoff, 2010; Blanchini & Miani, 2008; Girard, 2005; Kurzhanski & Varaiya, 2000, 2014; Le Guernic, 2009). Reachability analysis techniques that rely on ellipsoids (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Kurzhanski & Varaiya, 2000, 2007, 2014), level sets (Mitchell & Tomlin, 2000; Tomlin, Mitchell, Bayen, & Oishi, 2003), zonotopes (Althoff, Stursberg, & Buss, 2008; Girard, 2005; Girard, Le Guernic, & Maler, 2006), polytopes (Chutinan & Krogh, 2003), intervals (Lohner, 1987; Maïga, Ramdani, Travé Massuyès, & Combastel, 2016; Nedialkov, 2006; Ramdani, Meslem, & Candau, 2009), support functions (Le Guernic, 2009; Le Guernic & Girard, 2010), and numerical solution to ODEs (Duggirala & Viswanathan, 2016; Frehse, 2015) have been proposed. These methods have been applied to the analysis of aircraft (Tomlin et al., 2003), autonomous cars (Althoff, 2010; Althoff & Dolan, 2014), power systems (Althoff, 2014; Chen & Domínguez-García, 2012; El-Guindy, Han, & Althoff, 2016; Jin et al., 2005; Susuki et al., 2012), and wind turbines (Villegas Pico & Aliprantis, 2014, 2016). Computationally tractable reachability analysis techniques have been devised for linear dynamic systems (Althoff, 2010; Girard, 2005; Girard et al., 2006; Le Guernic, 2009). These approaches have been extended to hybrid, nonlinear, and differential algebraic systems, e.g., see Althoff (2010), Althoff (2014) and Dit Sandretto and Chapoutot (2016). Specialized methods to handle reachability analysis of discrete systems have been proposed as well (Blanchini & Miani, 2008; Raković & Fiacchini, 2008; Raković, Kerrigan, Mayne, & Lygeros, 2006).

A major thrust has been to develop numerically reliable tools for studying dynamic systems with uncertain initial conditions and parameters (Berz & Hoffstätter, 1998; Berz & Makino, 2002; CAPD Group, 2017; Dit Sandretto & Chapoutot, 2016; Lohner, 1987; Makino & Berz, 2006; Nedialkov, 2006). These tools can account for floating-point errors that arise in computations with finite machine precision (Gautschi, 2012). Such errors can lead to unreliable results when assessing, for example, *chaotic* dynamic systems with extreme sensitivity to initial conditions (Makino & Berz, 2006). Another variety of computational tools for reachability analysis deals with *linear* systems with uncertain initial conditions and inputs; these include, for example, State Space Explorer (SpaceEx) (Frehse et al., 2012) and Continuous Reachability Analyzer (CORA) (Althoff, 2016). These tools, however, do not account for floating-point errors (Althoff, 2016; SpaceEx, 2017), which is acceptable in the analysis of such systems.

One important missing feature from the present variety of reachability analysis techniques relying on polytopes for linear

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systems is that they do not reveal the inputs that drive state trajectories to the boundaries of the reachable set (Althoff, 2016; Frehse et al., 2012). As a common post-processing visualization step, deterministic simulations with heuristically selected initial conditions and time-varying inputs are conducted to illustrate containment within computed reachable sets (Althoff, 2016; Frehse et al., 2012); however, this approach is not rigorous. Recently, a technique has been proposed to generate state maximizer trajectories that underapproximate a reachable set as tightly as possible (Frehse, 2015). The method is based on support functions and the numerical solution of linear programs and ODEs, and it is computationally expensive. The particular inputs that yield the maximizer trajectories do not have a closed-form expression because they are generated numerically.

Knowing these worst-case inputs can be useful from a system design perspective. Depending on the application at hand, an input may have certain properties regarding its continuity or rate of change. For instance, a vehicle's steering wheel manipulation can be considered as an unknown-but-bounded, continuous, and, perhaps, slew rate-limited signal. In wind energy applications, an input could represent wind speed, which could be modeled as an unknown-but-bounded and continuous disturbance (Zárate-Miñano, Anghel, & Milano, 2013). These examples suggest a need to account for the continuity properties of the input to reflect the physical attributes of a system, thereby leading to more realistic reachable sets. For instance, consider the linear system

$$\frac{d}{dt}x(t) = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \tag{1}$$

for given sets of initial conditions $x(0) = \alpha \cdot (1/10, 0)^T$ with $\alpha \in [-1, 1] \subset \mathbb{R}$ and inputs $u : [0, T] \mapsto [-1, 1]$. The solution of (1) for any coefficient α and input u is

$$x(t) = \frac{\alpha}{10} \cdot \begin{pmatrix} e^{\sigma t}\cos(\omega t) \\ e^{\sigma t}\sin(\omega t) \end{pmatrix} + \int_0^t \begin{pmatrix} e^{\sigma(t-\tau)}\cos(\omega(t-\tau)) \\ e^{\sigma(t-\tau)}\sin(\omega(t-\tau)) \end{pmatrix} u(\tau)\, d\tau. \tag{2}$$

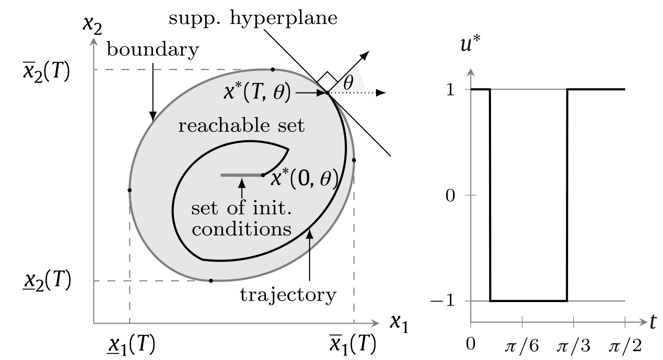
The reachable set at time $t = T$ considering all possible initial conditions and inputs is illustrated in Fig. 1. The sets of initial condition parameters and inputs that yield the boundary of the illustrated reachable set are

$$\alpha^*(T, \theta) = \mathrm{sgn}(\cos(\omega T - \theta)) \tag{3}$$

$$u^*(\tau; T, \theta) = \mathrm{sgn}(\cos(\omega(T - \tau) - \theta)),\ \tau \in [0, T] \tag{4}$$

for $\theta \in [0, 2\pi)$, respectively. [Theorem 7 of Section 3 leads to (3) and (4).] The angle θ represents the direction of a normal vector to a supporting hyperplane of the reachable set, as illustrated in Fig. 1. The set of input functions that drive the state trajectories to the boundary are square waves that cycle at ω rad/s. These discontinuous square-wave inputs may be physically unrealistic depending on the application, hence they are mostly of theoretical interest.

Calculating the exact reachable set boundary shown in Fig. 1 can be computationally expensive in higher dimensional linear systems. The calculation depends on the choice of set representation of the uncertain initial conditions and inputs (e.g., using ellipsoids, polytopes, zonotopes, or support functions). An alternative approach consists of calculating extrema of system states, that is, projections of the reachable set. For example, Fig. 1 illustrates the upper and lower bounds of the states $x_1(t)$ and $x_2(t)$ at $t = T$, so that $x_1(T) \in [\underline{x}_1(T), \overline{x}_1(T)]$ and $x_2(T) \in [\underline{x}_2(T), \overline{x}_2(T)]$. Knowing state time-domain bounds may suffice to warn a control engineer about potentially adverse system behavior. If the sets are represented by zonotopes (as done in this paper), this calculation is exact



(a) Reachable set. (b) Input.

Fig. 1. (a) A reachable set at time $T = \pi/2$ (gray area) with extrema of the states. (b) The deterministic input (black trace) that causes the state to reach the boundary at time T and at $\theta = \pi/4$ rad. System parameters are: $\sigma = -1$, $\omega = 4$.

(in the sense that it can be based on closed-form expressions) and computationally tractable.

Similar to other approaches, we rely on zonotope representations of the sets of initial conditions and inputs (Althoff, 2010; Girard, 2005). In prior work, zonotope generator expansion (growth of number of vectors Girard et al., 2006) and the wrapping effect (uncontrolled set expansion because of the propagation of recurring over-approximations Barbarosie, 1995; Girard et al., 2006) have been major computational hurdles. To overcome these, generator expansion has been limited by heuristic reduction techniques (Althoff, 2010; Althoff & Krogh, 2012; Girard, 2005; Girard et al., 2006), and the wrapping effect has been eliminated by employing special recursive algorithms (Althoff, 2010; Girard et al., 2006). However, in the proposed formulation, these problems are avoided because the technique hinges on closed-form expressions and does not rely on recursive set-based computations.

In view of the aforementioned issues, the contributions of this manuscript are: (i) A closed-form formulation for initial conditions and inputs that drive the states and outputs to the exact reachable set boundary, assuming that the sets of initial conditions and inputs are represented by zonotopes. To capture particular physical attributes of a given system, the inputs can be specified as either constant, completely arbitrary (possibly discontinuous), or bounded Lipschitz continuous (rate limited), the latter being the case of most interest and novelty in this work. (ii) Closed-form expressions that yield exact time-domain bounds of the states and outputs. This formulation helps expedite the calculations because the problem is reduced to evaluation of relatively simple functions. (iii) A computationally inexpensive method to calculate time-domain bounds. The technique relies on the zonotope representation of sets and a modal factorization of the system dynamics. Relatively simple integrals are obtained, with analytical expressions that are evaluated with floating-point precision (for the constant and arbitrary input cases) or numerically (for the Lipschitz continuous input case). Significant reductions in computational time compared to available tools (Althoff, 2016; Frehse et al., 2012) have been observed.

The paper is structured as follows. A concise introduction to modeling linear dynamic systems with uncertain initial conditions and inputs is presented in Section 2. Details of the underlying theoretical background are presented in Section 3. The numerical implementation of the technique is discussed in Section 4. Illustrative case studies are provided in Section 5. Section 6 concludes the manuscript.