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Brief paper

A lifting method for analyzing distributed synchronization on the unit sphere[☆]

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ABSTRACT

This paper introduces a new lifting method for analyzing convergence of continuous-time distributed synchronization/consensus systems on the unit sphere. Points on the d-dimensional unit sphere are lifted to the $(d+1)$-dimensional Euclidean space. The consensus protocol on the unit sphere is the classical one, where agents move toward weighted averages of their neighbors in their respective tangent planes. Only local and relative state information is used. The directed interaction graph topologies are allowed to switch as a function of time. The dynamics of the lifted variables are governed by a nonlinear consensus protocol for which the weights contain ratios of the norms of state variables. We generalize previous convergence results for hemispheres. For a large class of consensus protocols defined for switching uniformly quasi-strongly connected time-varying graphs, we show that the consensus manifold is uniformly asymptotically stable relative to closed balls contained in a hemisphere. Compared to earlier projection based approaches used in this context such as the gnomonic projection, which is defined for hemispheres only, the lifting method applies globally. With that, the hope is that this method can be useful for future investigations on global convergence.

1. Introduction

This paper considers systems of agents continuously evolving on $\mathbb{S}^{d-1}$, where $d \geq 2$. The interactions between the agents are changing as a function of time. For such systems we are analyzing a large class of distributed synchronization/consensus control laws. The analysis tool is a lifting method, where an equivalent consensus protocol is analyzed in the ambient space that embeds the sphere. In comparison to projection methods that have been used in this context – e.g., the gnomonic projection – the proposed method is not locally but globally defined on the unit sphere. The control action is performed in the tangent plane. Only relative information between neighboring agents is used in the control laws. Under the assumption that the time-varying graph is uniformly quasi-strongly connected, we show that the consensus manifold is globally uniformly asymptotically stable relative to any closed ball on the sphere contained in an open hemisphere.

Synchronization on the circle, i.e., $d = 2$, is closely related to synchronization of oscillators (Dörfler & Bullo, 2014) and it is equivalent to synchronization on $\mathsf{SO}(2)$, where several applications exist such as flocking in nature and alignment of multi-robot systems. Also for the two-dimensional sphere, i.e., $d = 3$, there are several applications such as formation flying and flocking of birds; consider for example a multi-robot system in 3D, where the relative directions between the robots are available and the goal is to align those. For higher dimensional spheres there are currently related problems such as distributed eigenvector computation, but concrete applications might arise in the future.

The control laws at hand – and slight variations or restrictions on the graph topologies, switchings of the graphs, dimensions of the sphere, and the nonlinear weights in the control laws etc. – have been studied from various perspectives (Li, 2015; Li & Spong, 2014; Olfati-Saber, 2006; Sarlette, 2009; Scardovi, Sarlette, & Sepulchre, 2007). There have recently been new developments (Markdahl & Goncalves, 2016; Markdahl, Wenjun, Hu, Hong, & Goncalves, 2016; Pereira & Dimarogonas, 2015; Pereira, Dimitris, & Dimarogonas, 2016). In Markdahl, Thunberg, and Gonçalves (2017), almost global consensus is shown by characterization of all equilibrium points when the graph is symmetric and constant

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(time-invariant). It is shown that the equilibria not in the consensus manifold are unstable and the equilibria in the consensus manifold are stable. A similar technique is used in Tron, Afsari, and Vidal (2012) to show that a consensus protocol on $\mathsf{SO}(3)$ is almost globally asymptotically stable. Now, the above-mentioned results about almost global convergence come at a price. Static undirected graph topologies are assumed as well as more restrictive classes of weights in the control protocols. Furthermore, compared to Markdahl et al. (2017), the right-hand sides of the system dynamics are not necessarily an intrinsic gradient and the linearization matrices at equilibria are not necessarily symmetric. Hence, we cannot use the result due to Lojasiewicz (1982) about point convergence for gradient flows. This inspired us to take a closer look at methods that transform the consensus problem on the unit sphere (or a subset thereof) to an equivalent consensus problem in $\mathbb{R}^d$. Before we address the method – referred to as a lifting method – we briefly make some connections to the related problem of consensus on $\mathsf{SO}(3)$.

The problem of consensus on $\mathsf{SO}(3)$ has been extensively studied (Deng, Liu, Wang, & Baras, 2016; Ren, 2010; Sarlette, Bonnabel, & Sepulchre, 2010; Sarlette, Sepulchre, & Leonard, 2009; Thunberg, Goncalves, & Hu, 2016; Tron, Afsari, & Vidal, 2013; Tron & Vidal, 2014). There is a connection between that problem and the problem of consensus on $\mathbb{S}^3$ when the unit quaternions are used to represent the rotations. For those, the gnomonic projection can be used to show consensus on the unit-quaternion sphere (Thunberg, Song, Hong, & Hu, 2014; Thunberg, Song, Montijano, Hong, & Hu, 2014). In another line of research, several methods have been introduced where control laws based on only relative information have been augmented with additional auxiliary (or estimation) variables, which are communicated between neighboring agents. By doing so, results about almost global convergence to the consensus manifold are achieved (Sarlette & Sepulchre, 2009; Thunberg, Markdahl, & Goncalves, 2017). The latter of these two publications provides a control protocol for Stiefel manifolds, with the unit sphere and $\mathsf{SO}(d)$ as extreme cases. A similar technique had previously been used for the sphere (Scardovi et al., 2007). The idea of introducing auxiliary variables also extends to the related distributed optimization problem in Thunberg, Bernard, and Goncalves (2017). In contrast to the mentioned works, in this paper we are not assuming additional communication between the agents by means of auxiliary variables. Instead only relative information is used in the protocols. In a practical setting (considering the case $d = 3$), such information can be measured by for example a vision sensor and requires no explicit communication between the agents.

In the proposed lifting method, we lift the states from the $(d-1)$-dimensional sphere into $\mathbb{R}^d$. The non-negative weights in the consensus protocol for the states in the lifting space are nonlinear functions. Each agent moves in a direction that is a weighted combination of the directions to the neighbors. The weights contain rational functions of the norms of the states of the agents. Since these rational functions are not well-defined at the origin, fundamental questions arise about existence, uniqueness, and invariance of sets. Those questions are answered with positive answers. The hope is that this lifting method will serve as a stepping-stone to future analysis on (almost) global convergence to the consensus manifold on the unit sphere. Compared to the approach in Markdahl et al. (2017) where all the "bad" equilibria on $\mathbb{S}^{d-1}$ were characterized, we only need to characterize one point, which is the origin in the "lifted space". If we were to show that this point has a region of attraction that is of measure zero, we would have equivalently shown the desired result about almost global convergence on the unit sphere (assuming $d \geq 3$). However, the non-differentiability of this point remains an additional challenge.

An extended version of this paper is available at arXiv, see Thunberg, Markdahl, Bernard, and Goncalves (2018).

2. Preliminaries

We begin this section with some *set-definitions*. The $(d-1)$-dimensional unit sphere is

$$\mathbb{S}^{d-1} = \{y \in \mathbb{R}^d : \|y\|_2 = 1\}.$$

The special orthogonal group in dimension d is

$$\mathsf{SO}(d) = \{Q \in \mathbb{R}^{d\times d} : Q^T = Q^{-1}, \det(Q) = 1\}.$$

The set of skew symmetric matrices in dimension d is

$$\mathsf{so}(d) = \{\Omega \in \mathbb{R}^{d\times d} : \Omega^T = -\Omega\}.$$

The set $\mathcal{H} \subset \mathbb{S}^{d-1}$ is an open *hemisphere* if there is $v \in \mathbb{S}^{d-1}$ such that $\mathcal{H} = \{w \in \mathbb{S}^{d-1} : w^T v > 0\}$.

We consider a multi-agent system with n agents. Each agent has a corresponding state $x_i(t) \in \mathbb{S}^{d-1}$ for $t \in [0, \infty)$. The initial state of each agent i at time 0 is $x_{i0} \in \mathbb{S}^{d-1}$. Another way to represent the states of the agents is to use rotation matrices. Let $R_i(t) \in \mathsf{SO}(d)$ satisfy $R_i(t)p = x_i(t)$ for all i and $t \in [0, \infty)$, where $p = [1, 0, \ldots, 0]^T$ is the *north pole*; we also define $-p$ as the *south pole*. Let $R_{i0}p = x_{i0}$ for all i, where R_{i0} is the initial R_i-matrix at time 0. The R_i-matrices can be interpreted as transformations from body coordinate frames – denoted by $\mathcal{F}_i$'s – of the agents to a world coordinate frame $\mathcal{F}_W$. They are transforming the unit vector p in the body frames to the corresponding unit vector (or point on the unit sphere) in the world coordinate frame. The R_i's and their dynamics are not uniquely defined, but this is not of importance for the analysis. We choose to define the dynamics of the R_i's according to (2) below.

The dynamics of the x_i-vectors are given by

$$\dot{x}_i = (I - x_i x_i^T)R_i[0, v_i^T]^T = R_i[0, v_i^T]^T, \tag{1}$$

where $v_i(t) \in \mathbb{R}^{d-1}$ for all t. The v_i-vectors are the controllers for the agents and those are defined in the body coordinate frames, i.e., the $\mathcal{F}_i$'s. For the R_i-matrices the dynamics is

$$\dot{R}_i = R_i \begin{bmatrix} 0 & -v_i^T \\ v_i & 0 \end{bmatrix}. \tag{2}$$

The matrix on the right-hand side of R_i in (2) is an element of $\mathsf{so}(d)$. The control is performed in the tangent space of the sphere, which means that there are $d-1$ degrees of freedom for the control. This is the reason why the v_i-vectors are $(d-1)$-dimensional. Before we proceed, we provide some additional explanation for the expression in the right-hand side of (2). According to its definition, the first column of R_i is equal to x_i and by multiplying $\dot{x}_i$ by R_i^T from the right we obtain – due to (1) –the following expression

$$R_i^T \dot{x}_i = [0, v_i^T]^T.$$

This means that

$$R_i^T \dot{R}_i = \begin{bmatrix} 0 & \star \\ v_i & \star \end{bmatrix},$$

where we the $\star$-parts are left to be chosen. We know that the matrix in the right-hand side above needs to be skew symmetric, since R_i is a rotation matrix. We also know that the first column of it must be equal to $[0, -v_i^T]^T$. The matrix of minimum Euclidean norm that fulfills these two requirements is equal to

$$\begin{bmatrix} 0 & -v_i^T \\ v_i & 0 \end{bmatrix},$$

i.e., the one we chose in the right-hand side of (2).

We will study a class of distributed synchronization/consensus control laws on the unit sphere, where the agents are moving in directions comprising conical combinations of directions to neighbors. In this protocol only local and relative information is