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Brief paper

A separation theorem for guaranteed H_2 performance through matrix inequalities[☆]

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ABSTRACT

The usage of convex optimisation programs that leverage linear matrix inequalities allows for numerical solutions to the design of output-feedback controllers with guaranteed H_2 performance. As decreed by the classical separation theorem for the related LQG control problem, the H_2 control problem admits an optimal solution in terms of those of the separate optimal state-estimation and state-feedback design problems. This work details a new and alternative proof of this separation theorem. The proof builds on techniques for (linear) matrix inequalities and shows, in particular, that feasible but sub-optimal solutions of the state-feedback and the state-estimation problem yield a sub-optimal output feedback controller with guaranteed H_2 performance.

1. Introduction

In control theory, the separation principle usually refers to a controller synthesis methodology in which a state estimator is designed independently from a state-feedback regulator so as to result in a controller that processes measurements to control inputs. A classic case is the linear quadratic Gaussian control problem for a linear time-invariant plant, in which an optimal controller is obtained as the interconnection of the Kalman filter that minimises the asymptotic covariance of a state estimation error and the optimal solution of a linear quadratic regulator problem. In this situation, the separation principle leads to an optimal controlled system, and one refers to the *separation theorem* instead of a principle.

There has been a consistent trend to perform computations for optimal control design in the realm of convex optimisation programs defined through constraints in terms of linear matrix inequalities (LMIs). Via interior point methods, these problems can be solved in polynomial time to any required precision. For example, the sub-optimal H_2 output-feedback control design problem can be phrased as a convex optimisation problem (Scherer, Gahinet, & Chilali, 1997), but until today it is restricted to lead to a full, that is unstructured, description of the controller without any separation property.

In this paper, we connect the feasibility of the matrix inequalities in the design problems for the estimator and state-feedback gain to the feasibility conditions of the fully parameterised H_2-output feedback problem. As a result, we obtain a new proof of the classical separation theorem that is only based on matrix inequality arguments. As a new feature, we show that any pair of sub-optimal solutions to the state-estimation and state-feedback control problem gives rise to a sub-optimal solution to the output feedback problem with a controller that admits a separation structure. This has particular relevance for multi-objective design problems where upper-bounds on achievable or realised performance are explored to meet additional design specifications. Indeed, separation of feasibility tests of the underlying convex optimisation problems may provide substantial insight and simplify the design process for these multi-objective controllers.

The separation theorem was first proven for continuous-time models by Wonham (1968). For discrete-time models, it was shown by using dynamic programming by Joseph and Tou (1961) and Striebel (1965). A more recent proof of separation by Davis and Zervos (1995) exploits Lagrange multipliers for quadratic performance objectives in the discrete-time case. The separation theorem for the existence and construction of H_2 optimal controllers, proven in Zhou, Doyle, Glover, et al. (1996), is a more delicate issue, as was addressed in Saberi, Sannuti, and Chen (1995).

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The paper is structured as follows. The main results are given in Section 4 and are preceded with the problem statement and background material in Sections 2 and 3. Conclusions are given in Section 5. In the remaining part of this section, we introduce the notation and recap some properties of the H_2-norm.

Notation and the H_2-norm. Denote the set of $n \times n$ real symmetric matrices as $\mathbb{S}^n$. If $A, B \in \mathbb{S}^n$, $A \prec (\preceq) B$ means that $A - B$ is negative (semi-)definite; this is equivalently expressed as $B \succ (\succeq) A$. If A is a square matrix, $\mathrm{tr}(A)$ is its trace, and A is said to be stable (Schur) if all its eigenvalues are contained in the open unit disk.

The discrete-time H_2 norm of the transfer matrix $T(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}$ with a stable matrix $\mathcal{A}$ (and with real matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of appropriate dimension) is given by

$$\|T\|_{H_2} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left[T(e^{j\omega})T(e^{j\omega})^*\right] \mathrm{d}\omega}. \tag{1}$$

If the realisation is minimal, then $\|T\|_{H_2}^2 = \mathrm{tr}(\mathcal{C}\mathcal{Y}\mathcal{C}^T)$ where $\mathcal{Y} \succ 0$ is the unique solution of the Stein equation $\mathcal{Y} - \mathcal{A}\mathcal{Y}\mathcal{A}^T - \mathcal{B}\mathcal{B}^T = 0$. Upper-bounds on the H_2 norm are characterised as follows (Scherer & Weiland, 2000).

Proposition 1. *For any $\lambda > 0$, the following statements are equivalent:*

(1) $\|T\|_{H_2} < \lambda$ *and $\mathcal{A}$ is stable;*

(2) *there exists a $\mathcal{Y} \succ 0$ such that*

$$\mathcal{Y} - \mathcal{A}\mathcal{Y}\mathcal{A}^T - \mathcal{B}\mathcal{B}^T \succ 0, \quad \sqrt{\mathrm{tr}(\mathcal{C}\mathcal{Y}\mathcal{C}^T)} < \lambda; \tag{2a}$$

(3) *there exists an $\mathcal{X} \succ 0$ such that*

$$\mathcal{X} - \mathcal{A}^T\mathcal{X}\mathcal{A} - \mathcal{C}^T\mathcal{C} \succ 0, \quad \sqrt{\mathrm{tr}(\mathcal{B}^T\mathcal{X}\mathcal{B})} < \lambda; \tag{2b}$$

(4) *there exist symmetric $\mathcal{Y}$ and Z such that*

$$\begin{bmatrix} \mathcal{Y} - \mathcal{B}\mathcal{B}^T & \mathcal{A}\mathcal{Y} \\ \mathcal{Y}\mathcal{A}^T & \mathcal{Y} \end{bmatrix} \succ 0, \begin{bmatrix} \mathcal{Y} & \mathcal{Y}\mathcal{C}^T \\ \mathcal{C}\mathcal{Y} & Z \end{bmatrix} \succ 0, \ \mathrm{tr}\, Z < \lambda^2; \tag{2c}$$

(5) *there exist symmetric $\mathcal{X}$ and Z such that*

$$\begin{bmatrix} \mathcal{X} - \mathcal{C}^T\mathcal{C} & \mathcal{A}^T\mathcal{X} \\ \mathcal{X}\mathcal{A} & \mathcal{X} \end{bmatrix} \succ 0, \begin{bmatrix} \mathcal{X} & \mathcal{X}\mathcal{B} \\ \mathcal{B}^T\mathcal{X} & Z \end{bmatrix} \succ 0, \ \mathrm{tr}\, Z < \lambda^2. \tag{2d}$$

2. Problem statement

Consider a system whose dynamics are given by the mathematical model

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + B_w w_t \\ y_t &= Cx_t + D_w w_t, \ z_t = C_z x_t + D_z u_t \end{aligned} \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $y_t \in \mathbb{R}^p$ is the measured output available for control, and $z_t \in \mathbb{R}^q$ is the (unmeasured) performance output. A, B, C are real matrices of appropriate dimensions. The system is subject to stochastic disturbances $w_t \in \mathbb{R}^{w_n}$ that affect the state transitions and measurements, modelled as a white noise sequence with a standard Gaussian distribution $\mathcal{N}\left(0, I_{w_n}\right)$.

Hypotheses on system matrices in (3).

(H1a) (A, B) is stabilisable and (C, A) is detectable;

(H1b) $D_z^T D_z \succ 0$ and $D_w D_w^T \succ 0$;

(H1c) $D_z^T C_z = 0$ and $D_w B_w^T = 0$.

(H2) (C_z, A) is observable and (A, B_w) is controllable.

All throughout the paper, we assume (H1a)–(H1c) to hold. Note that (H1c) is only introduced for notational brevity and can be easily removed. Some results on optimality are proven via Riccati equations and require (H2).

Controller and controlled system. Consider the set of controllers $\mathcal{K}$ with elements $\mathbf{K} \in \mathcal{K}$ defined as

$$x_{t+1}^{\mathbf{K}} = A_{\mathbf{K}} x_t^{\mathbf{K}} + B_{\mathbf{K}} y_t, \quad u_t^{\mathbf{K}} = C_{\mathbf{K}} x_t^{\mathbf{K}}. \tag{4}$$

The interconnection of system (3) with controller (4), defined by setting $u^{\mathbf{K}} = u$, yields the controlled system

$$\xi_{t+1} = \mathcal{A}\xi_t + \mathcal{B}w_t, \quad z_t = \mathcal{C}\xi_t \tag{5}$$

with

$$\mathcal{A} = \begin{bmatrix} A & BC_{\mathbf{K}} \\ B_{\mathbf{K}}C & A_{\mathbf{K}} \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B_w \\ B_{\mathbf{K}}D_w \end{bmatrix}, \ \mathcal{C} = [\, C_z \ D_z C_{\mathbf{K}} \,]. \tag{6}$$

Controller design. Consider the objective to choose controller parameters $(A_{\mathbf{K}}, B_{\mathbf{K}}, C_{\mathbf{K}})$ that render the controlled system (5) stable and minimise its H_2-norm. By Proposition 1, this can be expressed as

$$\begin{aligned} \lambda_{\inf} := \inf_{\lambda, \mathcal{Y}, \mathbf{K} \in \mathcal{K}} \lambda \quad \text{s.t.} \ & \sqrt{\mathrm{tr}(\mathcal{C}\mathcal{Y}\mathcal{C}^T)} < \lambda, \ \mathcal{Y} \succ 0, \\ & \mathcal{Y} - \mathcal{A}\mathcal{Y}\mathcal{A}^T - \mathcal{B}\mathcal{B}^T \succ 0 \end{aligned} \tag{7}$$

or, equivalently, as

$$\lambda_{\inf} := \inf_{\lambda, \mathcal{X}, \mathbf{K} \in \mathcal{K}} \lambda \qquad \text{s.t. (2b)}, \quad \mathcal{X} \succ 0. \tag{8}$$

Generally, the objective is twofold: one aims to compute the optimal value $\lambda_{\inf}$ and to find, if it exists, an optimal controller $\mathbf{K}^* \in \mathcal{K}$. A controller is optimal if, for each $\lambda > \lambda_{\inf}$, there exists some $\mathcal{Y}$ (respectively $\mathcal{X}$) for which the inequalities in (7) (respectively (8)) are satisfied. Observe that the order of the controller $\mathbf{K} \in \mathcal{K}$ is left free.

If viewed as a feasibility problem in $\mathcal{Y}$ and in the controller parameters $(A_{\mathbf{K}}, B_{\mathbf{K}}, C_{\mathbf{K}})$ for some fixed $\lambda > 0$, we refer to this controller synthesis problem as the *sub-optimal H_2 control problem*, which admits a numerically efficient solution through polynomial time methods (Masubuchi, Ohara, & Suda, 1998; Scherer et al., 1997) by conversion to a convex optimisation problem with constraints given as linear matrix inequalities. With Eq. (35) in Scherer et al. (1997), this conversion is based on a reparameterisation of the controller matrices, and the design algorithm leads to a sub-optimal H_2 controller without a particular structure.

In this work, we are interested in generating structured optimal controllers as they would intuitively result from using the separation principle. This means that the controller should be the composition of the *Kalman filter* and the *optimal static state-feedback controller*. In the sequel, we develop a separation theorem on the basis of matrix inequalities in which feasibility of the separated matrix inequalities allow us to define a controller with guaranteed H_2 performance. More precisely, we analyse under which conditions optimal or sub-optimal solutions of the *state-estimation problem* (see Section 3)

$$\inf_{Q \succ 0, L} \ \mathrm{tr}\ C_z Q C_z^T \tag{9}$$

$$\begin{aligned} \text{s.t.} \quad & Q - (A + LC)Q(A + LC)^T \\ & \quad - (B_w + LD_w)(B_w + LD_w)^T \succ 0 \end{aligned} \tag{10}$$

and of *the state-feedback problem* (see Section 3)

$$\inf_{F, P \succ 0} \ \mathrm{tr}\ B_w^T P B_w \tag{11}$$

$$\begin{aligned} \text{s.t.} \quad & P - (A + BF)^T P (A + BF) \\ & \quad - (C_z + D_z F)^T (C_z + D_z F) \succ 0 \end{aligned} \tag{12}$$

can be used to find an optimal or sub-optimal structured controller that solves the output-feedback problems in (7) or (8). We will only give the results for (7), since those for (8) follow trivially by duality. At this point, it is important to emphasise the well known fact that one can routinely convert (9)–(10) and (11)–(10) into semi-definite programs.