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Maximizing the smallest eigenvalue of a symmetric matrix: A submodular optimization approach☆

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ABSTRACT

This paper studies the problem of selecting a submatrix of a positive definite matrix in order to achieve a desired bound on the smallest eigenvalue of the submatrix. Maximizing this smallest eigenvalue has applications to selecting input nodes in order to guarantee consensus of networks with negative edges as well as maximizing the convergence rate of distributed systems. We develop a submodular optimization approach to maximizing the smallest eigenvalue by first proving that positivity of the eigenvalues of a submatrix can be characterized using the probability distribution of the quadratic form induced by the submatrix. We then exploit that connection to prove that positive-definiteness of a submatrix can be expressed as a constraint on a submodular function. We prove that our approach results in polynomial-time algorithms with provable bounds on the size of the submatrix. We also present generalizations to non-symmetric matrices, alternative sufficient conditions for the smallest eigenvalue to exceed a desired bound that are valid for Laplacian matrices, and a numerical evaluation.

1. Introduction

An increasingly widespread approach to controlling networked systems is to select a set of nodes to perform actuation (e.g., selecting generators to participate in power system control, or designating agents as leaders in multi-agent systems), while relying on network effects to steer the remaining nodes to a desired state (Liu, Slotine, & Barabási, 2011; Tanner, 2004). Mathematically, this approach is often modeled as creating an induced submatrix, in which rows and columns corresponding to the leaders are removed (Barooah & Hespanha, 2007). The dynamics of the remaining network nodes are then specified by the induced submatrix. A prominent example of this class of systems is the grounded Laplacian matrix, which is created in consensus networks when the states of a subset of leader nodes are set identically to zero (Pirani & Sundaram, 2014). Such matrices arise naturally in distributed estimation, formation control, and other multi-agent system problems.

The performance of such systems is known to be heavily influenced by the spectrum of the induced submatrix (Pirani & Sundaram, 2016). Of particular importance is the smallest eigenvalue of the induced sub-matrix. In Rahmani, Ji, Mesbahi, and Egerstedt (2009), it was shown that the rate of convergence of a consensus network is determined by the magnitude of the smallest eigenvalue of the grounded Laplacian matrix. The sign of the smallest eigenvalue determines whether the system is stable. Networks with antagonistic interactions, such as biological regulatory networks with repressive connections or social networks in which users disagree, may be unstable (Chen, Liu, Chen, Khong, Wang, Başar, Qiu, & Johansson, 2016; Meng, Shi, Johansson, Cao, & Hong, 2016; Zelazo & Bürger, 2017). Ensuring consensus in such systems is equivalent to selecting a submatrix in which all eigenvalues are positive.

These existing works show that the smallest eigenvalue of the grounded Laplacian matrix determines the stability and convergence of the networked system (Pirani & Sundaram, 2016; Rahmani et al., 2009; Zelazo & Bürger, 2017). Hence, an analytical approach to maximizing the smallest eigenvalue would lead to improved performance of such systems. However, so far to the best of our knowledge there are no computational techniques for maximizing the smallest eigenvalue. The main difficulty is that, unlike metrics such as the inverse trace (Clark, Bushnell, & Poovendran, 2014) and convergence error (Clark, Alomair, Bushnell, & Poovendran, 2014), the smallest eigenvalue of the grounded Laplacian is not known to possess any structure such as submodularity that enables development of efficient optimization algorithms with formal guarantees. Hence, while an efficient input selection algorithm with provable guarantees would improve the stability, robustness,

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and convergence rate of networked systems, at present no such algorithms that maximize the smallest eigenvalue are available.

In this paper, we present a submodular optimization approach to input selection in order to maximize the smallest eigenvalue of an induced submatrix such as the grounded Laplacian. Specifically, we investigate the problem of selecting a minimum-size input set in order to guarantee that the smallest eigenvalue is above a desired threshold. Our approach is as follows. We first prove that the eigenvalue condition holds if and only if an induced quadratic form is positive with probability one. Second, we show that this condition can be mapped to a constraint on a submodular function, equal to the probability that the quadratic form is zero when the input is a Gaussian random vector. Finally, we prove that this probability can be computed in polynomial time.

We analyze the optimality guarantees of our proposed approach and prove that the number of selected input nodes is within a logarithmic bound of the minimum-size input set. We show that the submodular optimization approach is applicable to problems including ensuring consensus of signed networks and maximizing convergence rate, and also explore generalizations to non-symmetric matrices (e.g., arising from directed graphs). We propose alternative sufficient conditions that are applicable to Laplacian matrices. Our sufficient conditions consist of bounds on the inverse trace and log determinant of the submatrix, and are shown to be submodular via spectral submodularity techniques. Our approach is validated through numerical study.

The paper is organized as follows. Section 2 reviews the related work. Section 3 gives relevant background. Section 4 presents the problem formulation and two motivating applications. Section 5 presents our proposed submodular framework. Section 6 discusses extensions to non-symmetric matrices and alternative sufficient conditions. Section 7 contains numerical results. Section 8 concludes the paper.

2. Related work

The importance of the smallest eigenvalue of grounded Laplacian graphs was identified in Rahmani et al. (2009), where it was shown that the magnitude of the smallest eigenvalue determines the rate of convergence to consensus. The eigenvalues of the grounded Laplacian were further studied in Pirani and Sundaram (2014, 2016). While these works analyzed the impact of the smallest eigenvalue and developed bounds on the smallest eigenvalue for different classes of graph, the problem of selecting nodes based on this criterion remains open.

Consensus in networks with both positive and negative edge weights, in which the negative weights represent antagonistic interactions between nodes, has been studied in Alemzadeh, de Badyn, and Mesbahi (2017) and Zelazo and Bürger (2014). Necessary and sufficient conditions for consensus in such networks without inputs based on effective resistance were proposed in Chen, Khong, and Georgiou (2016), Chen, Liu et al. (2016) and Zelazo and Bürger (2014). To the best of our knowledge, the only work that considers input selection in order to ensure consensus in networks with negative edges is the preliminary conference version of this paper (Clark, Hou, Bushnell, & Poovendran, 2017). Compared to Clark et al. (2017), this paper presents tighter necessary and sufficient conditions for consensus. The related problem of controllability of signed networks was proposed in Alemzadeh et al. (2017), but makes fundamentally different assumptions, namely that the input nodes can follow any arbitrary state trajectory.

The performance of networked systems with input nodes, often denoted as leaders, has been studied extensively (Jadbabaie, Lin, & Morse, 2003; Liu et al., 2011; Tanner, 2004). In particular, prior works have proposed techniques for selecting input nodes to optimize metrics including robustness to noise (Clark, Bushnell et al., 2014), convergence rate (Clark, Alomair et al., 2014), and controllability (Summers, Cortesi, & Lygeros, 2016), with submodular optimization as one approach. At present, however, there are no polynomial-time algorithms with provable guarantees for selecting input nodes in order to optimize the minimum eigenvalue of networked systems.

3. Notation and preliminaries

In what follows, we give necessary background on symmetric matrices, probability, and submodularity, and define notations that will be used throughout the paper.

Let I_n denote the $n \times n$ identity matrix. We omit the subscript n when the dimensionality of the matrix is clear for compactness of notation. A matrix A is symmetric if $A = A^T$, where A^T denotes the transpose of A. Any symmetric matrix can be written in the form $A = U \Lambda U^T$, where U is a unitary matrix (i.e., $UU^T = I$) and Λ is a real diagonal matrix. A symmetric matrix A is positive definite if all eigenvalues are positive, or equivalently, if $v^T A v > 0$ for all vectors v. The notation $A \succ 0$ denotes positive definiteness of A, while $A \succ B$ means that $(A - B)$ is positive definite. For any matrix A, the set of eigenvalues of A is denoted as $\lambda_1(A), \ldots, \lambda_n(A)$, where it is assumed that $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. We also use the notation $\lambda_{min}(A)$ to denote the minimum eigenvalue of A. Finally, we let $D(S)$ denote a diagonal matrix with $(D(S))_{ii} = 1$ if $i \in S$ and all other entries 0.

For an $n \times n$ matrix A, let $S \subseteq \{1, \ldots, n\}$ denote a set of indices. We let $A(S)$ denote the submatrix formed by the rows and columns indexed in S. Some interpretations of S and $A(S)$ are discussed in Section 4. The notation $A(S, T)$ refers to the submatrix with rows of A indexed in S and columns indexed in T. The following theorem describes the relationship between the eigenvalues of a matrix A and the eigenvalues of a submatrix.

Theorem 1 (*Cauchy Interlacing Theorem Horn & Johnson, 2012*). *Let A be an $n \times n$ symmetric matrix and let $A' = A(\{1, \ldots, n\} \setminus \{i\})$ for some $i \in \{1, \ldots, n\}$. Then*

$$\lambda_1(A) \geq \lambda_1(A') \geq \lambda_2(A) \geq \cdots \geq \lambda_{n-1}(A) \geq \lambda_{n-1}(A') \geq \lambda_n(A).$$

As a corollary to Theorem 1, we have that if $S \subseteq T \subseteq \{1, \ldots, n\}$, then $\lambda_{min}(A(S)) \leq \lambda_{min}(A(T))$, or in other words, the minimum eigenvalue $\lambda_{min}(S)$ is monotone increasing in the set S.

We now define notations and basic properties for certain random variables. Throughout the paper, we let $f_Z(z)$ and $F_Z(z)$ denote the probability density and distribution functions of random variable Z evaluated at $z \in \mathbb{R}$, respectively. We let $\mathbf{E}(\cdot)$ denote expectation, and let $Pr(\cdot)$ denote the probability of an event occurring.

Recall that for a Gaussian random vector $\mathbf{z}$, with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ, the random variable $M\mathbf{z}$ for any matrix M is Gaussian with mean $M\boldsymbol{\mu}$ and covariance $M \Sigma M^T$. If $X_1, \ldots, X_r$ are independent Gaussian random variables with zero mean and unit variance, then the random variable $Z = X_1^2 + \cdots + X_r^2$ is a chi-squared random variable with r degrees of freedom, with probability density function

$$f_Z(z) = \frac{z^{\frac{r}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}\right)}$$

for $z > 0$ and 0 otherwise, where Γ denotes the gamma function. The mean of Z is r, while the variance of Z is $2r$.

Finally, we give brief background on submodular functions. Let V denote a finite set. A function $f : 2^V \rightarrow \mathbb{R}$ that takes as input a subset of V and gives as output a real number is *submodular* if, for any sets $S, T \subseteq V$,

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T).$$