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Brief paper

Almost sure convergence of observers for switched linear systems ☆

Jianliang Tang [a], Mingqing Xiao [b,*]

[a] College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China
[b] Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, United States

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ABSTRACT

In this paper, we develop a new approach for the observer design of switched linear systems by ergodic theory and matrix analysis. Differently from most current results, we formulate the admissible switching sequences into a compact metric space and establish linear cocycles corresponding to the switched dynamics on this metric space under the framework of dynamical system theory. The necessary and sufficient condition for the almost surely convergent observer is given. Observer gains that minimize the unitarily invariant norm of the error dynamics during each switching process are obtained via pseudoinverse setting. An example is given to illustrate the proposed approach.

1. Introduction

A switched system consists of a family of subsystems associated with switching dynamics among subsystems. Transition among subsystems provides a practical modeling in many real applications, such as systems modeled for communication network, for robot manipulations, for traffic managements, etc. (e.g., see survey paper Liberzon & Morse, 2001, Lin & Antsaklis, 2009 and references therein). In addition, classical stochastic Markov jump systems can be viewed as switched systems associated with given transition probability distributions (see, e.g., Dai, Huang, & Xiao, 2015, Zhu, Yin, & Song, 2009 and references therein).

In this paper, we consider observer design of the switched linear system

$$x(k+1) = A_{\sigma(k)}x(k), \quad y(k) = C_{\sigma(k)}x(k), \tag{1}$$

where switchings are governed by a set of admissible mappings $\sigma : \mathbb{N} \to \{1, 2, \dots, N\}$, $N \in \mathbb{N}$, $N \geq 2$, $\mathbb{N}$ is the set of all natural numbers, $A_{\sigma(k)} \in \Sigma = \{A_1, A_2, \dots, A_N\} \subset \mathbb{R}^{n\times n}$, and the output $C_{\sigma(k)} \in \mathbb{R}^{m\times n}$ with $m < n$. Similar to classical observer design, we construct an observer through output injection:

$$\begin{aligned}\hat{x}(k+1) &= A_{\sigma(k)}\hat{x}(k) + K_{\sigma(k)}(\hat{y}(k) - y(k))\\ \hat{y}(k) &= C_{\sigma(k)}\hat{x}(k).\end{aligned} \tag{2}$$

The system consisting of (1) and (2) is called an extended (switched) system in literature, and all switching are among the subsystems of the extended system. By defining $\bar{x} = \hat{x} - x$, the error dynamics $\bar{x}$ satisfies

$$\bar{x}(k+1) = (A_{\sigma(k)} + K_{\sigma(k)}C_{\sigma(k)})\bar{x}(k). \tag{3}$$

Let us denote $\bar{A}_i = A_i + K_iC_i$, $i = 1, \dots, N$, and $\bar{\Sigma} = \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_N\}$. The observer design for (1) with a given set of admissible switchings is to seek K_i such that the absolute stability holds, i.e., for any initial condition $\bar{x}(0) = \bar{x}_0 \in \mathbb{R}^n$, , i.e. $\bar{x}(k) \to 0$ as $k \to \infty$. Differently from single systems, the study of the stability of error dynamics for a switched system is much harder and more challenging than the study of classical systems. The absolute stability of the error dynamics is characterized by the generalized spectral radius, defined as

$$\rho(\bar{\Sigma}) = \limsup_{\ell\to\infty} \max\left\{\rho(\bar{A}_{i_1}\cdots\bar{A}_{i_\ell})^{1/\ell} : \bar{A}_{i_s} \in \bar{\Sigma}\right\}$$

which was introduced by Daubechies and Lagarias in 1992 (Daubechies & Lagarias, 1992). If all matrices are identical, namely, $A_1 = A_2 = \cdots = A_N := A$, then $\rho(\bar{\Sigma}) = \rho(A)$, where $\rho(A)$ is the classical spectral radius of A (the largest absolute value of its eigenvalues). The absolute stability for the error dynamics (3) holds if and only if the generalized spectral radius $\rho(\bar{\Sigma}) < 1$. It describes the maximal asymptotic growth rate of all products of matrices taken from Σ.

It has been well-known that the generalized spectral radius is NP-hard to compute or to approximate, even when the set Σ consists of only two matrices and when all nonzero entries of the matrices are assumed to be equal, and the complexity of computations grows exponentially with respect to the required accuracy

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* Corresponding author.
E-mail addresses: jtang@szu.edu.cn (J. Tang), mxiao@math.siu.edu (M. Xiao).

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(see, e.g., Tsitsiklis & Blondel, 1997). Moreover, the question "$\rho \le 1$?" has been shown to be an undecidable problem (Blondel & Tsitsiklis, 2000). The essential challenge to compute $\rho(\Sigma)$ lies in (i) NP-hardness, it is valid even for switched binary matrices; (ii) undecidability, i.e., there does not exist in general any algorithm allowing to compute a joint spectral radius in finite time; (iii) non algebraicity, due to Kozyakin (1990), states that there is no algebraic criterion allowing to decide the stability of a switched linear system in general.

Many current approaches for the study of stability (as well as stabilization) of switched systems in literature focus on the absolute stability and are essentially based on the search of common control-Lyapunov functions or variations of similar framework, and the controlled stability usually is characterized by the existence of positive definite solutions of linear matrix inequalities (LMIs) (e.g., see survey paper Lin & Antsaklis, 2009). The goal is to seek a common control-Lyapunov function such that the energy of the overall dynamics decreases to zero along all feasible state trajectories governed by switches among subsystems (Liberzon & Tempo, 2004). Recent development under the Lyapunov function framework mainly includes (i) to seek the largest set of switching sequences for which the system is absolutely stable; or (ii) to determine the minimum dwell time such that a set of corresponding stabilizing switching sequences can be identified.

Although Lyapunov method is a common approach in current study of stability of switched systems, its drawback is obvious. It relies on the existence of a suitable common Lyapunov function, which usually is prior unknown and depends on an underlying given system. There is no guarantee that such a desirable Lyapunov function exists without imposing further restriction on the system structure (with the help of LMIs or similar), which quite often may lead to undesirable limitations of this type of approaches in applications. For switched systems, absolute stability may not be achievable even if each subsystem is asymptotically stable since switching dynamics also plays a critical role in the system stability. The instability can occur if the dynamics of any single admissible switching sequence is not stable. Thus to achieve the absolute stability usually requires quite strong assumptions on the systems that may not be desirable. On the other hand, almost sure stability (under appropriate probability measure) offers another option with much less restrictions (Dai, Huang, & Xiao, 2008, 2011; Dai et al., 2015) and is adequate in practical applications such as in the study of stochastic processes (Zhu et al., 2009).

The challenge to obtain the absolute stability of the error dynamics (3) is due to even if each pair (A_i, C_i) is observable and each $\rho(\bar{A}_i) < 1$, the absolute stability of (3) is not guaranteed in general. The requirement of $\rho(\bar{\Sigma}) < 1$ is much stronger than $\rho(\bar{A}_i) < 1$. To see that, without loss of generality, let us assume that the matrix set $\bar{\Sigma}$ is irreducible, then there exists a matrix norm $\|\cdot\|_*$, called extremal norm, defined on $\mathbb{R}^{n\times n}$ (Dai, Huang, & Xiao, 2013), such that

$$\rho(\bar{\Sigma}) = \max\left\{\|\bar{A}_1\|_*, \|\bar{A}_2\|_*, \ldots, \|\bar{A}_N\|_*\right\}.$$

Hence, $\rho(\bar{\Sigma}) < 1$ is equivalent to $\|\bar{A}_i\|_* < 1$ for all $i = 1, 2, \ldots, N$. The computation of the extremal norm is a forbidden task (equivalently, an NP-hard problem). This explains the challenges of absolute stability of switched systems: it is not only required the eigenvalues of $\bar{A}_i$ to be inside the unit disk, but also an existing matrix norm so that each $\|\bar{A}_i\|_* < 1$. Therefore pole placement is far more enough for the observer design of switched systems.

Motivated by our recent work (Dai et al., 2008), in this paper, we present a method of observer design for (1) whose observer is almost surely convergent in terms of Parry measure via symbolic dynamical theory. Parry measure is one of Markov measures and often used to capture the maximal set of stable processes under the stochastic framework (see, e.g., Walters, 1982). By using the matrix pseudoinverse, we construct the observer gain K_i and show that the error dynamics is minimized with all unitarily invariant norms during each switching.

The paper is organized as follows. In Section 2, we formulate the set of switching sequences into a compact metric space, associated with a transition probability matrix that resulted from switching constraints. The condition of almost surely stability of error dynamics under an ergodic Borel probability measure is presented. In Section 3, pseudoinverse is introduced to construct the observer gain and we discuss the minimization under unitarily invariant norms via matrix theory, which provides a better geometric view. Observer design under optimal unitarily invariant norm is presented in 4. The paper ends with concluding remarks in Section 5.

2. Formulation and almost sure stability

To study the stability of error dynamics (3), for arbitrary switching, we have to consider the stability of all possible infinite matrix products $\bar{A}_{i_1}\bar{A}_{i_2}\cdots\bar{A}_{i_j}\cdots$. Each product is associated with a (symbolic) switching sequence $[i_1 i_2 \cdots i_j \cdots] := [i]$, $1 \le i_j \le N$ ($N \ge 2$). Let us denote the set of all possible switching sequences by $\Lambda = \{[i] = [i_1 i_2, \ldots, i_\ell, \ldots], \text{ where } i_\ell \in \{1, 2, \ldots, N\}\}$, which is a set with infinite many sequences, and denote the one-sided shift by $\tau([i]) = \tau([i_1 i_2 \cdots i_j \cdots]) = [i_2 i_3 \cdots i_j \cdots]$. We further introduce a mapping:

$$S : \Lambda \to \{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N\} = \bar{\Sigma} \quad \text{by} \quad S([i]) = \bar{A}_{i_1}.$$

Then for any given switching sequence $[i]$, the error dynamics (3) with observer gain K_i can be written as $\bar{x}(k+1) = S(\tau^k([i]))\bar{x}(k)$, $\tau^0 := id$. It is well-known that Λ ($\cong \{1, 2, \ldots, N\}^{\mathbb{N}_0}$) is a Cantor set (Teschl, 2012), where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, which implies that the cardinality of the set of all switching sequences is the same as the one of classical Cantor set that is uncountable. By defining $d([i], [i']) = \sum_{j=1}^{\infty} |i_j - i'_j|/N^j$, $\forall [i], [i'] \in \Lambda, (\Lambda, \tau)$ forms a symbolic dynamical system.

In practice, quite often it is necessary to consider only certain subsets of Λ since it is possible that only some switching may be admissible. We define an $N \times N$ (0, 1)-matrix Γ as follows: if A_j is allowed to follow A_i ($1 \le i, j \le N$) then $(\Gamma)_{ij} = 1$ otherwise $(\Gamma)_{ij} = 0$. The corresponding subset of Λ is denoted by $\Lambda_\Gamma = \{[i] \in \Lambda \mid \Gamma_{i_k, i_{k+1}} = 1 \text{ for all } k \ge 1\}$. Clearly, Λ_Γ is a close subset of Λ. Without loss of generality, assume that Γ is irreducible and denote the spectral radius of matrix Γ by $\rho(\Gamma)$, according to Perron–Frobenius theorem (Horn & Johnson, 2013), there exist two positive vectors $u, v \in \mathbb{R}^N$ such that $\Gamma v = \rho(\Gamma)v$, $u^T\Gamma = \rho(\Gamma)u^T$, and $\sum_{i=1}^N u_i v_i = 1$. Next we define a transition probability matrix $P = (p_{ij})$ whose entries are set to be $p_{ij} = \frac{(\Gamma)_{ij} v_j}{\rho(\Gamma) v_i}$, $1 \le i, j \le N$. Denote $p = [p_1, p_2, \ldots, p_N]^T = [u_1 v_1, \ldots, u_N v_N]^T$. It is straightforward to check $p^T P = p^T$. Now one can define τ-invariant Markov measure μ on Λ_Γ (see, e.g., Walters, 1982) as $\mu([i_1 i_2, \ldots, i_\ell]) = p_{i_1} p_{i_1 i_2} \cdots p_{i_{\ell-1} i_\ell}$, where $[i_1 i_2, \ldots, i_\ell] := \left\{(j_1 j_2 \cdots j_\ell \cdots) \in \Lambda \mid j_1 = i_1, j_2 = i_2, \ldots, j_\ell = i_\ell\right\}$ is the cylinder defined by the word $(i_1 i_2, \ldots, i_\ell)$ of length $\ell(\ge 2)$. By the Kolmogorov Extension Theorem (Walters, 1982), this uniquely defines a measure μ which is an ergodic τ-invariant Borel probability measure on the set Λ_Γ (Dai et al., 2008, 2013), and it is also called the Parry measure of the topological Markov chain (Λ_Γ, τ). Notice that $S(\tau^k([i])) = S([i])S(\tau([i]))\cdots S(\tau^{k-1}([i]))$, which is a linear cocycle on (Λ_Γ, τ). By applying Kingman's subadditive ergodic theorem, for a given norm $\|\cdot\|$ of $\mathbb{R}^{n\times n}$, we have (see, e.g., Dai et al., 2008) $\lambda(\mu, S) := \lim_{k\to\infty} \frac{1}{k}\int_{\Lambda_\Gamma} \ln\|S(\tau^k([i]))\| d\mu([i]) = \inf_{k\in\mathbb{N}} \frac{1}{k}\int_{\Lambda_\Gamma} \ln\|S(\tau^k([i]))\| d\mu([i])$. The number $\lambda(\mu, S)$ is the Lyapunov exponent of S for the ergodic system $(\Lambda_\Gamma, \mu, \tau)$. According to the multiplicative ergodic theorem (Oseledets, 1968), for μ-a.e.