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Brief paper

# Dissipativity-based boundary control of linear distributed port-Hamiltonian systems ☆

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ABSTRACT

The main contribution of this paper is a general synthesis methodology of exponentially stabilising control laws for a class of boundary control systems in port-Hamiltonian form that are dissipative with respect to a quadratic supply rate, being the total energy the storage function. More precisely, general conditions that a linear regulator has to satisfy to have, at first, a well-posed and, secondly, an exponentially stable closed-loop system are presented. The methodology is illustrated with reference to two specific stabilisation scenarios, namely when the (distributed parameter) plant is in impedance or in scattering form. Moreover, it is also shown how these techniques can be employed in the analysis of more general systems that are described by coupled partial and ordinary differential equations. In particular, the repetitive control scheme is studied, and conditions on the (finite dimensional) linear plant to have asymptotic tracking of generic periodic reference signals are determined.



## 1. Introduction

Port-Hamiltonian systems (Maschke & van der Schaft, 1992) have been introduced to model lumped parameter physical systems in a unified manner, van der Schaft and Jeltsema (2014), and their generalisation to the infinite dimensional scenario led to the definition of distributed port-Hamiltonian systems (van der Schaft & Maschke, 2002), that turned out to be an effective framework for describing distributed parameter physical systems as boundary control systems (BCS) (Fattorini, 1968), i.e. as abstract systems whose dynamic is written in terms of a partial differential equation (PDE) with control and observation at the boundary of the domain. This paper aims at providing a general synthesis methodology of exponentially stabilising control laws for the class of linear BCS in port-Hamiltonian form extensively studied in Jacob and Zwart (2012) and Le Gorrec, Zwart, and Maschke (2005). In Le Gorrec et al. (2005), *all* the admissible inputs that let us to define a well-posed BCS are presented, together with a second similar parametrisation that characterises the (boundary) outputs. The distributed port-Hamiltonian system turns out to be dissipative (van der Schaft, 2000), with its Hamiltonian as storage function, and quadratic supply rate. In contrast with the generality of this result, the current researches on stabilisation techniques for distributed port-Hamiltonian systems (see e.g. Macchelli, 2013; Macchelli, Le Gorrec, Ramírez, & Zwart, 2017; Ramírez, Le Gorrec, Macchelli, & Zwart, 2014; Schöberl & Siuka, 2013; Villegas, Zwart, Le Gorrec, & Maschke, 2009), is focused on a particular input–output mapping: the BCS has to be in impedance form, i.e. input and output are selected so that the system is passive. Then, control design relies on passivity theory, and the most common strategy is to add dissipation at the boundary, and/or to shape the energy function to shift the equilibrium.

Since the idea is to determine general stability conditions for the class of BCS defined in Le Gorrec et al. (2005), passivity or the port-Hamiltonian structure are no longer required by the control system. At first, it is proved that the closed-loop system resulting from the feedback interconnection of a linear regulator and a BCS in port-Hamiltonian form is again a BCS if the controller is stable and dissipative with respect to a class of supply rates that is determined by the input–output mapping of the infinite dimensional plant. Moreover, the addition of dissipation makes the closed-loop storage function to decrease exponentially, thus implying exponential stability of the equilibrium. Thanks to these techniques, exponential stability is then proved for a large class

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of systems whose dynamic is described by coupled PDEs and ordinary differential equations (ODEs). This result is an extension of Ramírez et al. (2014), where exponential stability was proved under the hypothesis that the regulator is a strictly input passive port-Hamiltonian system, and that the BCS is in impedance form. However, it is important to underline that such an extension relies on some technical lemmas presented in Ramírez et al. (2014), and generalised here to cope with a larger class of BCS in port-Hamiltonian form.

The scenario that this paper defines can be summarised as follows. For any dynamical system resulting from the feedback interconnection of a BCS in port-Hamiltonian form Le Gorrec et al. (2005), and a stable, linear, dissipative, finite dimensional system, if a matrix inequality that involves the supply rates of both holds true, then the closed-loop system is well-posed, i.e. it defines a BCS in the sense of the semigroup theory (Curtain & Zwart, 1995, Definition 3.3.2), in which the input is the reference signal. Moreover, if the finite dimensional system is asymptotically stable and "enough dissipation" is added, then the closed-loop system is exponentially stable. The potentialities of this approach are at first illustrated in case the BCS is in impedance or in scattering form, van der Schaft (2000, Chapter 4.4.3), and sufficient conditions on the finite dimensional controller to have exponential stability in closed-loop are provided.

To illustrate how to apply the proposed methodology to the study of systems described by coupled PDEs and ODEs, the stability analysis of repetitive control (Hara, Yamamoto, Omata, & Nakano, 1988) in the linear case is presented. Repetitive control is a technique for tracking periodic exogenous signals with a known time period $T$, and its main properties depend on a particular element, the *repetitive compensator*, that consists of a time delay $T$ surrounded by a positive feedback loop: the regulator turns out to be a distributed parameter system, while the plant is finite dimensional. Once the repetitive compensator is written as a BCS in port-Hamiltonian form, we are able to determine under which conditions repetitive control schemes exponentially converge, thus assuring asymptotic tracking of generic periodic reference signals.

## 2. Distributed port-Hamiltonian systems

We refer to the class of port-Hamiltonian systems defined on real Hilbert spaces described by the PDE, Jacob and Zwart (2012) and Le Gorrec et al. (2005) :

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + (P_0 - G_0)\mathcal{L}(z)x(t,z) \tag{1}$$

with $x \in X := L^2(a,b;\mathbb{R}^n)$, $z \in [a,b]$, and $\mathcal{L} : [a,b] \to \mathbb{R}^{n\times n}$ a bounded and Lipschitz continuous function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z) > 0$ for all $z \in [a,b]$. Since $\mathcal{L}$ is a coercive operator, $X$ is then endowed with the inner product $\langle x_1 \mid x_2\rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2\rangle$ and norm $\|x_1\|^2_{\mathcal{L}} = \langle x_1 \mid x_1\rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot\rangle$ denotes the natural $L^2$-inner product. The selection of this space for the state variable is motivated by the fact that $\|\cdot\|^2_{\mathcal{L}}$ is strongly linked to the energy function of (1). As a consequence, $X$ is also called the space of energy variables, and $(\mathcal{L}x)(t,z) := \mathcal{L}(z)x(t,z)$ denote the co-energy variables. Moreover, $P_1$, $P_0$ and $G_0$ are $n \times n$ real matrices, with $P_1 = P_1^{\mathrm{T}}$ and invertible, $P_0 = -P_0^{\mathrm{T}}$, and $G_0 = G_0^{\mathrm{T}} \geq 0$.

**Remark 1.** The PDE (1) can be written as $\dot{x} = \mathcal{J}x$, where $\mathcal{J}x := P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ is a linear operator with domain $D(\mathcal{J}) = \{\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)\}$, being $H^1(a,b;\mathbb{R}^n)$ the Sobolev space of order one. Note that $\langle \mathcal{J}x \mid x\rangle_{\mathcal{L}} = -\langle \mathcal{L}x \mid G_0\mathcal{L}x\rangle + e_\partial^{\mathrm{T}} f_\partial \leq e_\partial^{\mathrm{T}} f_\partial$, where

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix}. \tag{2}$$

The vectors $f_\partial$, $e_\partial \in \mathbb{R}^n$ are defined as linear combination of the restriction of $\mathcal{L}x \in H^1(a,b;\mathbb{R}^n)$ to the boundary. The problem of characterising inputs and outputs for (1) in terms of $f_\partial$ and $e_\partial$ to have a BCS on $X$ in the sense of the semigroup theory (Curtain & Zwart, 1995, Definition 3.3.2) has been addressed in Le Gorrec et al. (2005) when $G_0 = 0$. The case $G_0 \geq 0$ is a straightforward extension of the results presented in Le Gorrec et al. (2005), and it is discussed in the next theorem.

**Theorem 2.** *Let $W$ be a full rank $n \times 2n$ real matrix, and define $\mathcal{B} : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$ and the input $u(t)$ as*

$$u(t) = \mathcal{B}x(t) := W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}, \tag{3}$$

*with $D(\mathcal{B}) = D(\mathcal{J})$. The operator $\bar{\mathcal{J}}x := P_1 \frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain*

$$D(\bar{\mathcal{J}}) = \left\{\mathcal{L}x \in H^1(a,b;\mathbb{R}^n) \mid \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \operatorname{Ker} W\right\}$$

*generates a contraction semigroup on $X$, see* Curtain and Zwart (1995, Definition 2.2.1)*, if and only if*

$$W\Sigma W^{\mathrm{T}} \geq 0, \text{ where } \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

*and the system (1) with input (3) is a BCS on $X$, see* Curtain and Zwart (1995, Definition 3.3.2)*, provided that $u \in C^2(0,\infty;\mathbb{R}^n)$. Moreover, let $\tilde{W}$ be a full rank $n \times 2n$ matrix such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, and define the output as*

$$y(t) = \mathcal{C}x(t) := \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}, \tag{4}$$

*with $\mathcal{C} : H^1(a,b;\mathbb{R}^n) \to \mathbb{R}^n$. Then, for $(\mathcal{L}x)(0) \in H^1(a,b;\mathbb{R}^n)$, and $u(0) = \mathcal{B}x(0)$, the following energy-balance inequality is satisfied:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{L}} \leq \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} \underbrace{\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-\mathrm{T}} \Sigma \begin{pmatrix} W \\ \tilde{W} \end{pmatrix}^{-1}}_{=:P_{W,\tilde{W}}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}. \tag{5}$$

**Proof.** In Le Gorrec et al. (2005, Theorem 4.1), it has been proved that the operator $J_W e := P_1 \frac{\partial e}{\partial z} + P_0 e$ with domain

$$D(J_W) = \left\{ e \in L^2(a,b;\mathbb{R}^n) \mid R\begin{pmatrix} e(b) \\ e(a) \end{pmatrix} \in \operatorname{Ker} W \right\}$$

is the infinitesimal generator of a contraction $C_0$-semigroup $(T(t))_{t\geq 0}$ on $X$. As a consequence, see e.g. Jacob and Zwart (2012, Theorem 10.3.1), $J_W - G_0$ with the same domain is the infinitesimal generator of a $C_0$-semigroup $(T_G(t))_{t\geq 0}$ on $X$, which is again a contraction because $T_G(t) = e^{-G_0 t}T(t)$ and $G_0 = G_0^{\mathrm{T}} \geq 0$. So, the first statement of the theorem follows in the same way as in Le Gorrec et al. (2005, Lemma 5.4) since $\bar{\mathcal{J}} = (J_W - G_0)\mathcal{L}$. The operator $\bar{\mathcal{J}}$ defines the dynamics of (1) when the input $u(t)$ selected as in (3) is equal to zero. Existence of solutions in the autonomous case is one of the requirements for having a BCS in the sense of the semigroup theory, Curtain and Zwart (1995, Definition 3.3.2). The other ones can be checked in the same way as in steps 2 and 3 of the proof of Le Gorrec et al. (2005, Theorem 4.2). Finally, the balance relation (5) is a consequence of Remark 1, because $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2_{\mathcal{L}} = \langle \mathcal{J}x \mid x\rangle_{\mathcal{L}}$, which implies that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2_{\mathcal{L}} \leq e_\partial^{\mathrm{T}} f_\partial$, and of the definitions (3) and (4) of $u$ and $y$ in terms of $f_\partial$ and $e_\partial$.

**Corollary 3.** *The BCS of Theorem 2 is dissipative* (van der Schaft, 2000) *with storage function $E(x) = \frac{1}{2}\|x\|^2_{\mathcal{L}}$, and supply rate*

$$s(u,y) =: \frac{1}{2}\begin{pmatrix} u \\ y \end{pmatrix}^{\mathrm{T}} \underbrace{\begin{pmatrix} U & S \\ S^{\mathrm{T}} & Y \end{pmatrix}}_{:=P_{W,\tilde{W}}} \begin{pmatrix} u \\ y \end{pmatrix}, \tag{6}$$