Automatica 95 (2018) 146–151

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation[☆]

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ARTICLE INFO

Article history:
Received 21 July 2017
Received in revised form 3 January 2018
Accepted 16 April 2018

Keywords:
G-Brownian motion
Asymptotic stability
Mean-square exponential stability
Feedback control
Discrete-time state observation

ABSTRACT

This paper mainly concerns the stability of the solutions for stochastic differential equations driven by G-Brownian motion (G-SDEs) via feedback control based on discrete-time state observation. More precisely, the discrete-time state feedback control is included in the drift coefficient of the G-SDEs. By constructing an appropriate G-Lyapunov function, a set of conditions is obtained for the H_∞ stability, asymptotic stability and mean-square exponential stability of the controlled systems. Finally, an example with numerical simulation is presented to illustrate the effectiveness of the proposed control design technique.

1. Introduction

Since Peng (see, e.g., Peng, 2007, 2008) set up the G-expectation and G-Brownian motion theory, many interesting works have been done on stochastic calculus based on G-Brownian motion due to its powerful applications in uncertain problems, risk measures, the superhedging in finance and so on. Under the G-framework, stochastic differential equations driven by G-Brownian motion (G-SDEs) were first introduced by Peng (see, e.g., Peng, 2008). Since then, some interesting works have been reported on the qualitative properties such as existence, uniqueness and stability of the solutions for G-SDEs; for more details, one can see, e.g., Li, Lin, & Lin (2016); Ren, Jia, & Sakthivel (2017) and the references therein. Especially, Li et al. (see, e.g., Li et al., 2016) have proposed the sufficient conditions for the exponential instability to the following G-SDE:

$$\begin{aligned} \mathrm{d}x(t) = {} & f(t,x(t))\mathrm{d}t + h(t,x(t))\mathrm{d}\langle B\rangle(t) \\ & + \sigma(t,x(t))\mathrm{d}B(t), \quad t \geq 0. \end{aligned} \tag{1}$$

Specifically, natural question is whether we can design a control to make the G-SDE (1) stable.

There are many methods to make an unstable system stable. Among them, the feedback control based on discrete-time state observation is an efficient one, which has been used in establishing the mean-square exponential stabilization for a class of continuous time hybrid stochastic differential equations driven by Brownian motion (see, e.g., Mao, 2016). Since then, many authors have discussed the stability for different kinds of stochastic dynamical systems (see, e.g., Hu & Mao, 2008; Mao, 2013, 2016; Mao, Lam, & Huang, 2008; Mao, Liu, Hu, Luo, & Lu, 2014; Mao, Yin, & Yuan, 2007; Qiu, Liu, Hu, & Lu, 2016; Qiu, Liu, Hu, Mao, & You, 2016; Shao, 2017; You, Liu, Liu, Mao, & Qiu, 2015). In particular, for an unstable stochastic system, it is necessary and important to design a feedback control with the form $u(t, x([t/\tau]))$ embedded into the drift part, where $[t/\tau]$ is the integer part of t/τ, where τ is the discrete-time observation gap.

Motivated by the aforementioned works, it is necessary to develop a controller which can make G-SDEs (1) stable. The main aim of this work is to find a feedback control based on discrete-time state observation $u(t, x([t/\tau]\tau))$ embedded into the drift coefficient, so that the controlled system

$$\begin{aligned} \mathrm{d}x(t) = {} & [f(t,x(t)) + u(t,x([t/\tau]\tau))]\mathrm{d}t + h(t,x(t))\mathrm{d}\langle B\rangle(t) \\ & + \sigma(t,x(t))\mathrm{d}B(t), \quad t \geq 0 \end{aligned} \tag{2}$$

becomes stable. In particular, the H_∞ stability, asymptotic stability and mean-square exponential stability of the controlled system are

☆ This work is supported by the National Natural Science Foundation of China (11371029 and 11501009) and the Natural Science Foundation of Anhui Province (1508085JGD10). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Oswaldo Luiz V. Costa under the direction of Editor Richard Middleton.

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https://doi.org/10.1016/j.automatica.2018.05.039
0005-1098/

obtained by means of G-Lyapunov function. The main contributions of this paper are summarized as follows:

- In this work, an unstable stochastic differential equation driven by G-Brownian motion is stabilized via feedback control based on discrete-time state observation.
- By constructing an appropriate G-Lyapunov function, a set of conditions can be derived for obtaining the required result.
- The result reveals that the designed controller makes the unstable system as a stable one with an adequate level of discrete-time observation gap.

In the last section, an example is proposed to illustrate the obtained results.

2. Problem formulation and preliminaries

In this section, we consider an n-dimensional controlled system in the following form:

$$\begin{aligned} dx(t) &= [f(t,x(t)) + u(t,x(\delta_t))]dt + h(t,x(t))d\langle B\rangle(t) \\ &\quad + \sigma(t,x(t))dB(t),\ t \geq 0 \end{aligned} \tag{3}$$

with initial data $x(0) = x_0 \in \mathrm{R}^n$, where $B(\cdot)$ is a one dimensional G-Brownian motion, $\langle B\rangle(\cdot)$ is the quadratic variation process of the G-Brownian motion $B(\cdot)$. Here

$$f, h, \sigma : \mathrm{R}^+ \times \mathrm{R}^n \to \mathrm{R}^n \ \text{ and } f, h, \sigma \in M_G^2(0,T),$$

while $\tau > 0$ and $\delta_t = [t/\tau]\tau$, where $[t/\tau]$ is the integer part of t/τ, τ is the discrete-time observation gap.

Noted that system (3) is in fact a G-SDE with a bounded variable delay $\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - l\tau \ \text{ for } l\tau \leq t < (l+1)\tau,\ \ l = 0,1,2,\ldots.$$

Then, the system (3) can be rewritten as

$$\begin{aligned} dx(t) &= [f(t,x(t)) + u(t,x(t-\zeta(t)))]dt \\ &\quad + h(t,x(t))d\langle B\rangle(t) + \sigma(t,x(t))dB(t). \end{aligned} \tag{4}$$

It can be easily seen that Eq. (4) has a unique solution $x(t)$ for any initial value x_0. For stability analysis, it is assumed that $x_0 = 0, f(t,0) = 0, h(t,0) = 0, \sigma(t,0) = 0, u(t,0) = 0$ for all $t \geq 0$, then system (3) admits a trivial solution $x(t) \equiv 0$.

Let $C^{1,2}(\mathrm{R}^+ \times \mathrm{R}^n; \mathrm{R}^+)$ be the family of all nonnegative functions $V(t,x)$ on $\mathrm{R}^+ \times \mathrm{R}^n$, which once differentiable are continuous in t and twice differentiable in x. If $V \in C^{1,2}(\mathrm{R}^+ \times \mathrm{R}^n; \mathrm{R}^+)$, then define an operator LV by

$$\begin{aligned} LV(t,x) &:= V_t(t,x) + V_x(t,x)[f(t,x) + u(t,x)] \\ &\quad + G(\langle V_x(t,x), 2h(t,x)\rangle \\ &\quad + \langle V_{xx}(t,x)\sigma(t,x), \sigma(t,x)\rangle), \end{aligned}$$

where $V_t(t,x) = \frac{\partial V(t,x)}{\partial t}$, $V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right)$, $V_{xx}(t,x) = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}$.

In order to stabilize the system (1), we impose the following assumptions.

Assumption 1. Suppose that the functions f, h and g are continuous and satisfy the following linear growth condition:

$$|f(t,x)| \leq L_1|x|,\ |h(t,x)| \leq L_2|x|,\ |\sigma(t,x)| \leq L_3|x|, \tag{5}$$

for all $(t,x) \in \mathrm{R}^+ \times \mathrm{R}^n$, where L_1, L_2 and L_3 are positive constants.

Assumption 2. Suppose that the controller function $u : \mathrm{R}^+ \times \mathrm{R}^n \to \mathrm{R}^n$ is globally Lipschitz continuous such that

$$|u(t,x) - u(t,y)| \leq L_4|x-y|,$$

where L_4 is a positive constant.

Assumption 3. Suppose that there exist two positive numbers λ_1, λ_2 such that $LV(t,x) + \lambda_1|V_x(t,x)|^2 \leq -\lambda_2|x|^2$ for all $(t,x) \in \mathrm{R}^+ \times \mathrm{R}^n$.

In this paper, we construct a G-Lyapunov functional on the segment $x_t := \{x(t+r) : -\tau \leq r \leq 0\}$ for $t \geq 0$. For x_t to be well defined for $0 \leq t \leq \tau$, we set $x(r) = x_0$ for $-\tau \leq r \leq 0$. Let

$$\begin{aligned} \bar{V}(t,x_t) &= V(t,x(t)) + \theta \int_{t-\tau}^{t}\int_{r}^{t} [\tau|f(v,x(v)) \\ &\quad + u(v,x(\delta_v))|^2 + \bar{\sigma}^4\tau|h(v,x(v))|^2 \\ &\quad + \bar{\sigma}^2|\sigma(v,x(v))|^2]dvdr,\ \ t \geq 0, \end{aligned} \tag{6}$$

where θ is a positive constant to be determined later. Applying the G-Itô formula to $\bar{V}(t,x_t)$, we have

$$d\bar{V}(t,x_t) = L\bar{V}(t,x_t)dt + dM_t, \tag{7}$$

where M_t is a G-martingale. Moreover,

$$\begin{aligned} L\bar{V}(t,x_t) &= V_t(t,x(t)) + V_x(t,x(t))[f(t,x(t)) \\ &\quad + u(t,x(\delta_t))] + G(\langle V_x(t,x(t)), 2h(t,x(t))\rangle \\ &\quad + \langle V_{xx}(t,x(t))\sigma(t,x(t)), \sigma(t,x(t))\rangle) \\ &\quad + \theta\tau\big[\tau|f(t,x(t)) + u(t,x(\delta_t))|^2 \\ &\quad + \bar{\sigma}^4\tau|h(t,x(t))|^2 + \bar{\sigma}^2|\sigma(t,x(t))|^2\big] \\ &\quad - \theta\int_{t-\tau}^{t} [\tau|f(r,x(r)) + u(r,x(\delta_r))|^2 \\ &\quad + \bar{\sigma}^4\tau|h(r,x(r))|^2 + \bar{\sigma}^2|\sigma(r,x(r))|^2]dr. \end{aligned} \tag{8}$$

Applying G-Itô formula to $V(t,x(t))$, we have

$$\begin{aligned} dV(t,x(t)) &= (V_t(t,x(t)) + V_x(t,x(t))[f(t,x(t)) \\ &\quad + u(t,x(\delta_t))])dt + G(\langle V_x(t,x(t)), 2h(t,x(t))\rangle \\ &\quad + \langle V_{xx}(t,x(t))\sigma(t,x(t)), \sigma(t,x(t))\rangle) + dM_t. \end{aligned}$$

Applying the fundamental theory of mathematical analysis, we get

$$\begin{aligned} &d\Big(\int_{t-\tau}^{t}\int_{r}^{t} \theta[\tau|f(v,x(v)) + u(v,x(\delta_v))|^2 + \\ &\bar{\sigma}^4\tau|h(v,x(v))|^2 + \bar{\sigma}^2|\sigma(v,x(v))|^2]dvdr\Big) \\ &= \Big(\theta\tau\Big[\tau|f(t,x(t)) + u(t,x(\delta_t))|^2 + \bar{\sigma}^4\tau|h(t,x(t))|^2 \\ &+ \bar{\sigma}^2|\sigma(t,x(t))|^2\Big] - \theta\int_{t-\tau}^{t}[\tau|f(r,x(r)) + u(r,x(\delta_r))|^2 \\ &+ \bar{\sigma}^4\tau|h(r,x(r))|^2 + \bar{\sigma}^2|\sigma(r,x(r))|^2]dr\Big)dt. \end{aligned}$$

Consequently, (8) holds. For some basic notations about G-Brownian motion, lemmas and definitions, one can refer the papers (Peng, 2007, 2010).

Definition 1. For any $\eta_t \in M_G^{1,0}(0,T)$, $\langle B\rangle(t)$ is the quadratic variation process of $B(t)$, define

$$\int_0^T \eta_t d\langle B\rangle(t) := \sum_{j=0}^{N-1} \xi_j\left(\langle B\rangle(t_{j+1}) - \langle B\rangle(t_j)\right),$$

where $\langle B\rangle(t) := \lim_{N\to\infty}\sum_{j=0}^{N-1}\left(B(t_{j+1}^N) - B(t_j^N)\right)^2 = B^2(t) - 2\int_0^t B(s)dB(s)$.

Proposition 1 (Peng, 2007). *For all $\eta_t \in M_G^2(0,T)$, we have $\hat{\mathbb{E}}\left(\int_0^T \eta_t dB(t)\right) = 0$ and $\hat{\mathbb{E}}\left(\int_0^T \eta_t dB(t)\right)^2 = \hat{\mathbb{E}}\left(\int_0^T \eta_t^2 d\langle B\rangle(t)\right) \leq \bar{\sigma}^2\hat{\mathbb{E}}\left(\int_0^T \eta_t^2 dt\right)$.*